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## A multiple indicators model for volatility using intra-daily data

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### Abstract

Many ways exist to measure and model financial asset volatility. In principle, as the frequency of the data increases, the quality of forecasts should improve. Yet, there is no consensus about a “true” or “best” measure of volatility. In this paper we propose to jointly consider absolute daily returns, daily high–low range and daily realized volatility to develop a forecasting model based on their conditional dynamics. As all are non-negative series, we develop a multiplicative error model that is consistent and asymptotically normal under a wide range of specifications for the error density function. The estimation results show significant interactions between the indicators. We also show that one-month-ahead forecasts match well (both in and out of sample) the market-based volatility measure provided by the VIX index as recently redefined by the CBOE.

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## 1. Introduction

Models to describe and predict financial asset volatility abound. In practice, in addition to a model's capability to reproduce stylized facts in observed time series and exhibit desirable statistical properties, the ultimate way to evaluate a model is its usefulness as a tool in many areas such as derivative products pricing, risk evaluation and hedging, portfolio allocation, and the derivation of value at risk measures.

Yet, the concept of volatility itself is somewhat elusive, as many ways exist to measure it and hence to model it (cf. the survey by Andersen et al., 2002). In recent times, the availability of ultra-high frequency data and the work done on them has shed new light on the concept of volatility: as a matter of fact, data sampled at regular intra-daily intervals can be summarized into a measure called realized volatility which under some assumptions is a consistent estimator of the quadratic variation of the underlying diffusion process. Such a measure was widely adopted as a target of forecast accuracy, but the dependence of the measure upon the frequency of observation of the data makes it difficult to come to clear conclusions. Moreover, as shown by Oomen (2002) such a measure may be biased if returns used to compute it are serially correlated. In principle, the volatility measures derived from ultra-high frequency data should prove to be more accurate, hence allowing for forecast efficiency gains. Nevertheless, the measures that make more intensive use of such data are prone to all sorts of microstructure problems. Bid-ask bounce (Roll, 1984), screen fighting (Zhou, 1996), price discreteness, irregular spacing of quotes and transactions can all bias volatility estimates.

Even if the growing literature on realized volatility has delivered promising results (Andersen et al., 2000; Barndorff-Nielsen and Shephard, 2002), what is of interest is the appropriate way to provide accurate forecasts in the medium-to-long run, and the problem remains open as to whether daily or intra-daily models deliver the most successful answer. In the first category one can mention those models favoring the existence of long memory or high persistence in the process of volatility, such as the daily component model by Engle and Lee (1999), or the fractionally integrated GARCH (FIGARCH—Baillie et al., 1996; or FIEGARCH, Bollerslev and Mikkelsen, 1996) or long memory stochastic volatility models (LMSV—Breidt et al., 1998; Deo and Hurvich, 1999). Intra-daily models are more recent: Ghose and Kroner (1996) adopt a signal plus noise model to estimate a persistent component as in Engle and Lee (1999); Andersen and Bollerslev (1997) show how accuracy of volatility forecasts can be improved if one moves to analyzing five-min returns; Engle (2000) derives a measure of volatility from transaction data.

The approach which we will pursue here is one in which several measures of volatility can be jointly used to see whether different features of observed time series can deliver an enrichment of volatility forecasting for the medium run. As a matter of fact, next to the traditional volatility modelling from daily returns measured as the log-difference of closing prices, we can consider absolute returns on which considerable modeling effort is present in the literature (Taylor, 1986; Ding et al., 1993; Granger and Sin, 2000) and, with the already mentioned provisos, realized volatility as the standard deviation of intra-daily returns observed at regular

intervals. Furthermore, it has long been recognized that the spread between the highest recorded daily price and the lowest recorded daily price is a function of the volatility during the day and, as such, can lead to an improvement of the volatility estimates. Many authors (Taylor, 1987; Rogers and Satchell, 1991; Gallant et al., 1999; Chou, 2001; Alizadeh et al., 2002; Brandt and Jones, 2002; Brandt and Diebold, 2003) have devoted considerable attention to the informational content of range data extending the relationship it has to the volatility parameter in a geometric Brownian motion context (cf. the early papers by Parkinson, 1980; Garman and Klass, 1980; Beckers, 1983), and comparing its persistence characteristics with the ones of daily returns (Brunetti and Lildholdt, 2003).

It should be stressed that the three variables have different features relative to one another: the main difference is that the daily return uses information about the closing price of the previous trading day, while the high–low spread and the realized volatility are measured on the basis of what is observed during the day, the former taking all trade information into account, the latter being built on the basis of quotes sampled at discrete intervals. Thus, a zero return is not necessarily informative about what happened during the day, and, by the same token, a high return may signal high volatility during the day while it may just be due to an opening price much different from the closing price the previous day but very close to the closing price of the same trading day, with a small high–low spread. Also, note that the same value of realized volatility may correspond to different values of the range since we could have recorded both high and low values fairly far apart during the day with a smooth transition of price movements between the two, or some vivacious price swings concentrated in a short period of time. For these reasons, they are all potentially useful and it becomes an empirical question whether they are indeed relevant and which interactions one may expect among them.

In the present paper we exploit the fact that absolute returns, daily range and realized volatility are variables evolving as functions of the underlying time-varying volatility and all exhibit the usual conditional persistence of financial time series. They can each be considered as indicators of volatility and since they are all non-negative-valued they can be modelled with a multivariate extension of the multiplicative error model suggested by Engle (2002). Each indicator is modeled as a GARCH-type process possibly augmented with weakly exogenous variables. The derived estimator is robust to a range of error distribution assumptions. We show that this three-variable model possesses interesting properties in that the forecasts for each indicator are augmented by the presence of the other indicators lagged and by asymmetric effects from the direction of price movements. The model can be solved dynamically for multi-period forecasts and the dynamic interdependence accounts for a substantial departure from the standard GARCH profile of dynamic forecasts. We can use our three equation model to predict multi-step volatility.

The model is estimated with daily data for the S&P500 stock index over a relatively long sample period. A careful specification search selects models for each equation. We calculate 22-step volatility forecasts and compare these with one month option-implied volatilities as measured by the volatility index VIX (in its new

definition). We examine the incremental explanatory power of our forecasts (both in and out of sample) over a simple autoregressive specification of VIX. The motivation for such an approach is to provide evidence of the relevance of model based forecasts such as the ones derived in our framework in explaining the behavior of a market based volatility measure such as the volatility index. One may argue that the volatility index should also be inserted as a weakly exogenous variable in the conditional variance equation for the corresponding index as done by Blair et al. (2001), or that even VIX should be seen as an indicator and a model for it added; in either case one would lose the evidence of model based forecasts being able to track a series that was not inserted in the information set at the estimation stage which is what we want to focus on.

The reader should expect the following: in Section 2 we discuss the Multiplicative Error Model, with some general considerations about the estimator and its properties, and about the specifications of the three models adopted. Data issues and model selection procedures are described in detail in Section 3. Model performance is analyzed in Section 4 focusing on the characteristics of the model in multi-step-ahead forecasting. In order to show the usefulness of our forecasts when compared to a measure based on implied volatilities representing market evaluation of volatility, in Section 5 we build multi-step-ahead volatility forecasts and we use them as additional regressors over a simple autoregressive specification for the volatility index VIX. Simple significance testing assesses the relevance of the regressors in their incremental explanatory power. Concluding remarks follow.

## 2. The model

### 2.1. General discussion

We assume that the evolution of a non-negative valued process  $x_t$ , can be described by a multiplicative error model (MEM, as discussed by Engle, 2002). That is,  $x_t$  is the product of a time varying scale factor (which depends upon the recent past of the series) and a standard positive valued random variable. Such a specification was adopted in Engle and Russell (1998) for durations, Manganelli (2002) for volume transaction data and Chou (2001) for high–low range. The scale factor is identified as the conditional mean if the error distribution is assumed to have unit mean. In general, therefore,

$$x_t = \mu_t \varepsilon_t, \quad \varepsilon_t | \mathfrak{F}_{t-1} \sim i.i.d.D(1, \varphi), \quad (1)$$

where  $\mu_t$  can be rather flexibly specified as

$$\mu_t = \omega + \sum_{i=1}^p \alpha_i x_{t-i} + \sum_{j=1}^q \beta_j \mu_{t-j} + c' z_t. \quad (2)$$

Further terms can signal the dependence of the series on weakly exogenous variables (summarized in the vector  $z_t$ ) included in the information set available at time  $t-1$ .

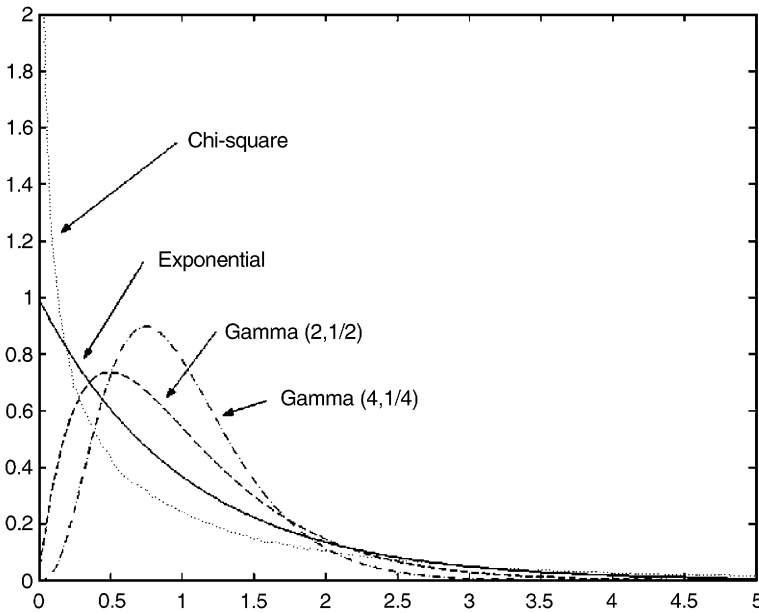


Fig. 1. Comparison among different gamma densities subject to the constraint  $ab = 1$ .

The conditions to ensure stationarity and positive means for all possible realizations have been discussed in Engle (2002).

The density of the error term  $\varepsilon_t$  was left unspecified thus far. While in general one should specify the true DGP, it may be possible to find robust specifications. We now consider the family of gamma densities that have been used for ACD models.

$$f(\varepsilon_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(a)b^a} \varepsilon_t^{a-1} \exp\left(-\frac{\varepsilon_t}{b}\right).$$

The property of unit mean<sup>1</sup> implies that  $b = 1/a$ , namely,

$$f(\varepsilon_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(a)} a^a \varepsilon_t^{a-1} \exp(-a\varepsilon_t) \Rightarrow f(x_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(a)} a^a x_t^{a-1} \mu_t^{-a} \exp\left(-a \frac{x_t}{\mu_t}\right). \tag{3}$$

The process would have conditional expectation  $E(x_t | \mathfrak{F}_{t-1}) = \mu_t$  and conditional variance  $Var(x_t | \mathfrak{F}_{t-1}) = \mu_t^2/a$ . A comparison between various densities corresponding to different choices of  $\alpha$  subject to the constraint is reported in Fig. 1. As well known, the choice of  $a = 1/2$ , gives the pdf of a chi-square with one degree of freedom, while  $a = 1$  yields the unit exponential. In general, values of  $a < 1$  amount to attributing more weight to extremely small or large values of the random variable

<sup>1</sup>Such a condition is not restrictive at all: consider that if  $\varepsilon_t$  were to be such that  $E(\varepsilon_t) = ab \neq 1$ , it could be written as  $b^* \varepsilon_t^*$  with  $E(\varepsilon_t^*) = 1$  and  $b^*$  would be a multiplicative constant to be absorbed by  $\mu_t$ .

while values of  $a > 1$  generate a hump shaped density which approaches the normal distribution for large values of  $a$ .

Let us suppose for the moment that the only parameters of interest are the ones defining  $\mu_t$  (let us call them  $\theta$ ), and let us look at the corresponding log-likelihood function: the relevant object for estimation is given by

$$\log L = \text{constant} - a \sum_{t=1}^T \left( \log \mu_t + \frac{x_t}{\mu_t} \right), \tag{4}$$

where the constant depends only on the  $x_t$ 's and  $a$ . It is clear that in maximizing this function with respect to  $\theta$ , the value assumed by  $a$  is irrelevant, as the first order conditions must obey

$$\sum_{t=1}^T \left( \frac{x_t - \mu_t}{\mu_t^2} \right) \frac{\partial \mu_t}{\partial \theta} = 0. \tag{5}$$

Even from a numerical point of view, an iterative procedure is unaffected (in a Newton-type method the  $a$  terms present in the inverse Hessian and gradient would cancel each other) resulting in exactly the same estimates (apart from rounding off errors when numerical derivatives are used). Furthermore, these first order conditions coincide with the ones derived from an auxiliary model that specifies the square root of the variable of interest,  $\sqrt{x_t}$ , as the product of the square root of the scale factor  $\sqrt{\mu_t}$  and a half Gaussian error term  $v_t$ , namely,

$$\sqrt{x_t} = \sqrt{\mu_t} v_t \quad v_t | \mathfrak{F}_{t-1} \sim \text{half Gaussian (standard)}. \tag{6}$$

Notice that this corresponds to the Gaussian GARCH model where  $x_t$  is the squared return,  $\mu_t$  is the conditional variance, and the error term  $v_t$  is a chi square with one degree of freedom.

This model has the clear advantage of being able to exploit any GARCH software which can estimate the parameters  $\theta$  in  $\mu_t$  from a model for  $\sqrt{x_t}$  with no equation for the mean (as done for the ACD model by [Engle and Russell, 1998](#); cf. also [Engle, 2002](#)).

Clearly, the second order conditions would differ, since the Hessian will be proportional to  $a$  (e.g. for the exponential case it would be twice the Hessian for the chi-square), and, correspondingly, the estimated parameter variance–covariance matrices. Note, however, that (exploiting the results of [Bollerslev and Wooldridge, 1992](#); [Lee and Hansen, 1994](#)) the robust variance–covariance matrix, computed as the product

$$\widehat{Var}(\hat{\vartheta}) = \widehat{H}^{-1} \widehat{OPG} \widehat{H}^{-1}$$

(with  $H$  the Hessian matrix and  $OPG$  the matrix of the outer products of the gradients) provides the obvious benefit of making this discrepancy irrelevant since the  $a$ 's cancel out:

$$\widehat{Var}(\hat{\vartheta}) = \frac{1}{a} \widehat{H}_{a=1}^{-1} a^2 \widehat{OPG}_{a=1} \frac{1}{a} \widehat{H}_{a=1}^{-1} = \widehat{H}_{\text{exp}}^{-1} \widehat{OPG}_{\text{exp}} \widehat{H}_{\text{exp}}^{-1}. \tag{7}$$

In the absence of justifications for the most appropriate distribution to adopt for the error term in an MEM, the strong lesson we learn from this discussion is that the most straightforward way of deriving its parameter values is to estimate an auxiliary variance equation for the positive square root of the variable of interest<sup>2</sup> with a GARCH specification and normally distributed errors. The estimators are quasi maximum likelihood estimators, hence consistent and asymptotically normal as discussed by Engle (2002), building on results by Lee and Hansen (1994). However, if we know that a Gamma distribution assumption for  $\varepsilon_t$  is appropriate, then the same procedure delivers consistency and efficiency for the estimators if  $a$  is known. Even if  $a$  is not known, using robust standard errors shields against the specific shape of the Gamma distribution. Introducing  $a$  among the parameters to be estimated would provide information on the shape of the distribution of the error term  $\varepsilon_t$  (some empirical results are discussed by Engle, 2002) which would be useful for simulating future values or scenario analysis but would not have any impact on the values of the estimates of  $\theta$ , nor on their standard errors as  $Cov(\hat{\theta}, \hat{a}) = 0$ .

### 2.2. Model specification for the volatility indicators

More specifically for the case at hand here, let us indicate the daily closing price as  $C_t$ , and calculate the daily returns as  $r_t = \log(C_t/C_{t-1})$ . Let us thus consider squared returns  $r_t^2$ , modeled as an MEM

$$r_t^2 = h_t^r \varepsilon_t, \quad \varepsilon_t | \mathfrak{F}_t \sim i.i.d.D(1, \xi^r); \tag{8}$$

where  $h_t^r$  is the conditional mean of  $r_t^2$ . The square,  $hl_t^2$ , of the high–low range is defined as  $hl_t = \log(H_t/L_t)$ , where  $H_t$  and  $L_t$  are the highest, respectively, the lowest recorded prices during the day, and is modeled as:

$$hl_t^2 = h_t^h \eta_t, \quad \eta_t | \mathfrak{F}_{t-1} \sim i.i.d.D(1, \xi^h); \tag{9}$$

and the square of the realized volatility,  $v_t$ , defined as the square root of the sum of squared returns over  $J$  subperiods within the day, between market opening time and market closing time,<sup>3</sup>

$$v_t^2 = h_t^v \zeta_t, \quad \zeta_t | \mathfrak{F}_{t-1} \sim i.i.d.D(1, \xi^v). \tag{10}$$

<sup>2</sup>As established by Ding et al. (1993), absolute returns exhibit high levels of serial correlation and can themselves be adopted as the first indicator of volatility. We prefer to think of the square root of  $r_t^2$  of absolute returns even if the choice between writing (6) for absolute returns or for returns (with Gaussian innovations) is irrelevant from a numerical point of view, since the first-order conditions would not change.

<sup>3</sup>The merits of the choice of the appropriate  $J$  to balance estimation accuracy and microstructure pitfalls are discussed in Andersen et al. (2000): empirically, 5 min intervals “work” in a foreign exchange framework (Andersen et al., 2001), 15 min intervals are adopted by Schwert (1998), but we still lack a theory of what is the optimal length to choose. In this context we choose  $J = 78$  and we adopted the method discussed by Zhou (1996) on raw data to purge bid-ask bounce effects.

Put together, we can write this trivariate MEM as a system of equations

$$\begin{pmatrix} r_t^2 \\ hl_t^2 \\ v_t^2 \end{pmatrix} = \begin{pmatrix} h_t^r \\ h_t^h \\ h_t^v \end{pmatrix} \odot \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \zeta_t \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \zeta_t \end{pmatrix} \Big| \mathfrak{F}_{t-1} \sim i.i.d.D(\mathbf{1}, \Xi) \tag{11}$$

where  $\odot$  indicates the Hadamard product,  $\mathbf{1}$  is a unit  $(3 \times 1)$  mean vector, and  $\Xi$  is a  $3 \times 3$  variance–covariance matrix of the innovation terms. The multivariate estimation problem becomes a series of univariate problems when it is assumed that the covariance matrix is diagonal. There is no loss of consistency if this assumption is false but there can be a loss of efficiency.

We can now turn to the details of the MEM specification. Let us consider the first MEM and write the basic form of the model for the scale factor  $h_t^r$  as

$$h_t^r = \omega_r + \alpha_r r_{t-1}^2 + \beta_r h_{t-1}^r. \tag{12}$$

In what follows, we will refer to this as the *base specification*, which is maintained even if individual coefficients turn out to be not significant in order to maintain identifiability of the parameters.

In order to take asymmetric reactions to shocks, this base model can be expanded, either by writing a threshold-type model involving the customary dummy variable for negative returns  $d_t = I(r_t < 0)$ ,

$$h_t^r = \omega_r + \alpha_r r_{t-1}^2 + \beta_r h_{t-1}^r + \gamma_r r_{t-1}^2 d_{t-1} \tag{13}$$

or even adding lagged returns  $r_{t-1}$  as a different way of accounting for asymmetry (as in the APARCH model, Ding et al., 1993) given that they maintain their sign,

$$h_t^r = \omega_r + \alpha_r r_{t-1}^2 + \beta_r h_{t-1}^r + \gamma_r r_{t-1}^2 d_{t-1} + \delta_r r_{t-1}. \tag{14}$$

The main question we want to address at this stage is whether the inclusion of the lagged variables  $hl_{t-1}^2$  and  $v_{t-1}^2$ , which are part of the information set  $\mathfrak{F}_{t-1}$ , adds significant explanatory power to the specification for  $h_t^r$ . If so, then some of the information contained in these indicators relates to a variability in returns which cannot be accounted for by considering just squared returns and possible asymmetric effects. By the same token, a way to account for further asymmetric effects when returns in the previous day are negative is to include also the product of the indicators by the same dummy variable  $d_{t-1}$ . A model which would take all these effects into account can be written as

$$\begin{aligned} h_t^r = & (\omega_r + \alpha_r r_{t-1}^2 + \beta_r h_{t-1}^r) + \gamma_r r_{t-1}^2 d_{t-1} + \delta_r r_{t-1} \\ & + \varphi_r hl_{t-1}^2 + \vartheta_r hl_{t-1}^2 d_{t-1} + \psi_r v_{t-1}^2 + \lambda_r v_{t-1}^2 d_{t-1}, \end{aligned} \tag{15}$$

where a subscript “r” was added to the coefficients indicating that they refer to the specification for returns. The augmentation of the GARCH(1,1) model in this case includes the six variables  $r_{t-1}$ ,  $r_{t-1}^2 d_{t-1}$ ,  $hl_{t-1}^2$ ,  $hl_{t-1}^2 d_{t-1}$ ,  $v_{t-1}^2$ ,  $v_{t-1}^2 d_{t-1}$  which are all known at time  $t-1$ . In what follows we will refer to this general form as the *system specification* (with whatever variables the chosen model selection will retain, see below).



A similar approach can be followed for the other two indicators. Considering again  $hl_t^2 = h_t^h \eta_t$ , and picking six relevant variables from  $\mathfrak{S}_{t-1}$ , namely,  $r_{t-1}$ ,  $hl_{t-1}^2 d_{t-1}$ ,  $r_{t-1}^2$ ,  $r_{t-1}^2 d_{t-1}$ ,  $v_{t-1}^2$ ,  $v_{t-1}^2 d_{t-1}$ , one can write a full specification for  $h_t^h$  as

$$h_t^h = (\omega_h + \alpha_h hl_{t-1}^2 + \beta_h h_{t-1}^h) + \delta_h r_{t-1} + \gamma_h hl_{t-1}^2 d_{t-1} + \varphi_h r_{t-1}^2 + \vartheta_h r_{t-1}^2 d_{t-1} + \psi_h v_{t-1}^2 + \lambda_h v_{t-1}^2 d_{t-1}. \tag{16}$$

Analogously, for the realized daily volatility indicator we start from  $v_t^v = h_t^v \zeta_t$  and add the corresponding six variables,  $v_{t-1}^2 d_{t-1}$ ,  $r_{t-1}$ ,  $r_{t-1}^2$ ,  $r_{t-1}^2 d_{t-1}$ ,  $hl_{t-1}^2$ ,  $hl_{t-1}^2 d_{t-1}$ , to the base specification of a GARCH(1,1) to get

$$h_t^v = (\omega_v + \alpha_v v_{t-1}^2 + \beta_v h_{t-1}^v) + \delta_v r_{t-1} + \gamma_v v_{t-1}^2 d_{t-1} + \varphi_d r_{t-1}^2 + \vartheta_v r_{t-1}^2 d_{t-1} + \psi_v hl_{t-1}^2 + \lambda_d hl_{t-1}^2 d_{t-1}. \tag{17}$$

The question that we will discuss in the following section is based on the characteristics of the data at hand, namely, whether the system specification is substantially different from the base specification, i.e. whether adding the lagged indicators adds substantial explanatory power. In fact, it may happen that the MEM for one indicator does not require the information set to be augmented, while there may be significant effects for another.

### 3. The data, parameter estimation and model selection

For our empirical models, we will use data on the Standard and Poor 500 index from January 4, 1988 to December 14, 1998 (2730 observations); we leave the last 217 observations for out-of-sample comparison purposes. We build series for absolute returns, high–low range and realized volatility (on five minute intervals) for this period: since the constructed variables have quite different scales relative to one another, we rescale all three variables so that they are all expressed in annual percentage terms and they share the in-sample quadratic mean of observed returns. Some descriptive statistics are reported in Table 1 and suggest that the absolute returns and the daily range have more characteristics in common with one another than each of them does with realized volatility. However, a more complete picture is obtained if one looks also at the correlation coefficients between the three series reported in the last lines of Table 1 and the scatter-plots in Fig. 2.

The absolute returns lie almost invariably below the daily range, signaling that the days in which the previous day closing price is lower (higher) than the lowest (highest) intradaily price are infrequent. A more dispersed pattern is shown between the absolute returns and the realized volatility as reflected also by the correlation coefficient of 0.51. The fact that the correlations are relatively high but not nearly unity signifies that the indicators are different from one another. These preliminary stylized facts coupled with the characteristics of persistence exhibited by the absolute return, the daily range and the realized volatility series (cf. Fig. 3), mean that an effort aimed at modeling the interactions between these three variables in a conditional context seems promising. In-sample there is a date, October 28, 1997,

Table 1  
S&P500 absolute returns, daily range and realized volatility. Summary statistics

	Absolute returns	Daily range	Realized volatility
Mean	9.35	11.27	7.74
Median	6.72	9.67	4.89
Maximum	112.92	84.83	241.53
Minimum	0.000	0.098	0.004
Std. Dev.	9.29	6.83	10.67
Quadratic mean	13.18	13.18	13.18
Skewness	2.93	2.90	7.60
Kurtosis	21.97	21.48	115.80
Correlations			
With absolute returns		0.81	0.51
With daily range			0.80

All variables are expressed in terms of percentage annual terms and they share the same quadratic mean. Sample period: January 4, 1988–December 30, 1997.

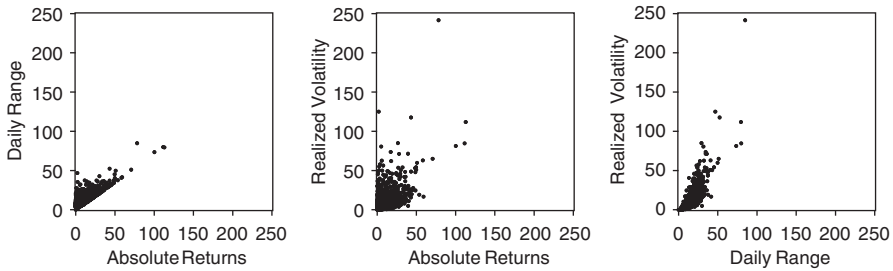


Fig. 2. The S&P500 index. Scatter-plots between absolute returns, daily range and realized volatility. Sample period: January 4, 1988–December 31, 1997.

which serves well as an example of what different indicators may record.<sup>4</sup> At the height of the turmoil period following the Asian currency crisis, on that day the S&P500 opened at 873.10, from a previous close of 876.98, and closed at 921.25 after having recorded a low at 855.31 and a high at 923.09. It is not surprising that the realized volatility in such a day turned out to be so high: one may think of a limit case in which the index bounces between the daily high and the daily low every 5 min; the realized volatility would be much higher than the daily range, and this gives account also of the puzzling evidence that the kurtosis of the realized volatility is so high.

<sup>4</sup>Just to make sure that the results would not be driven by such an abnormal observation, we rerun the estimation substituting the in-sample average to the observation for that day: the results changed only marginally.

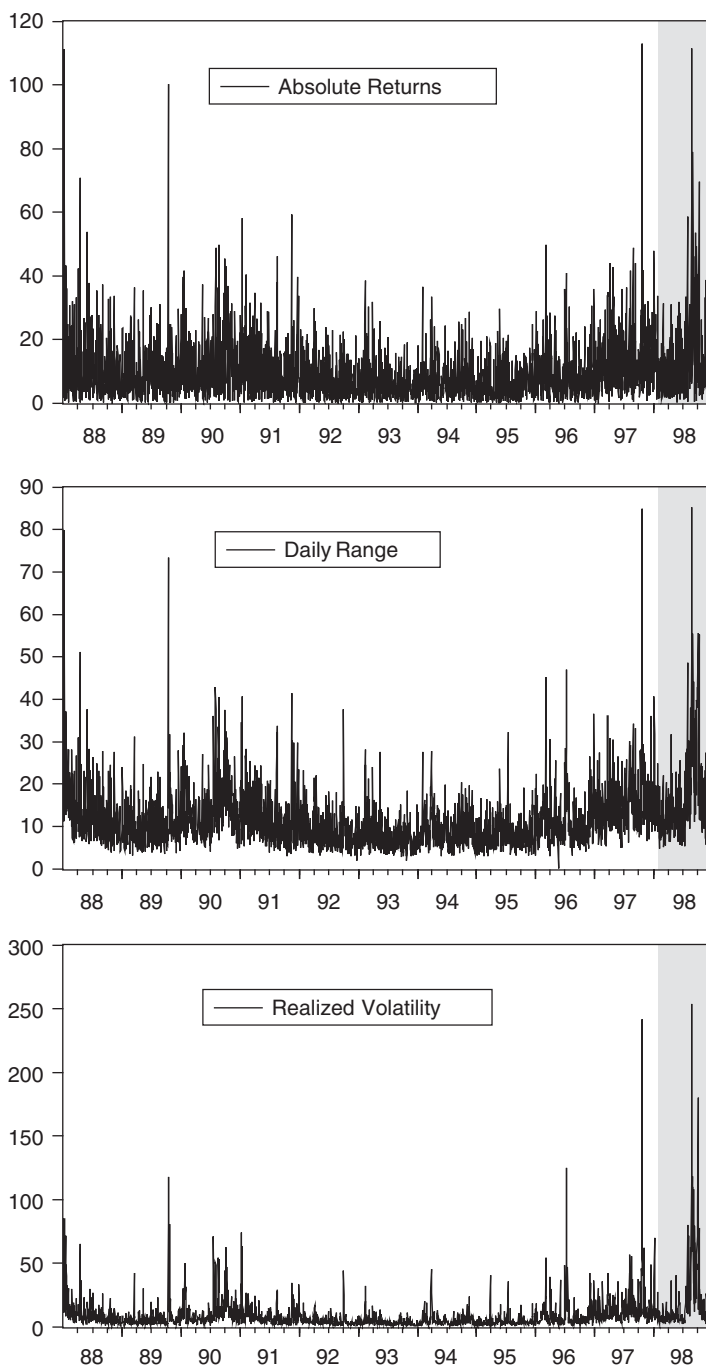


Fig. 3. The S&P500 index: absolute returns, daily range, and realized volatility 1/4/1988–12/14/1998 (out of sample period is shaded).

As discussed in Section 2, for each indicator we envisage the introduction of any of up to six weakly exogenous variables in addition to the GARCH(1,1)-type base specification. We are thus estimating  $2^6 = 64$  models ranging from the most general forms (15)–(17) down to the base specification kept as a benchmark; although in the latter some coefficients may not be significant, they will be kept in order for the model to be identified.

The two model selection strategies that we will adopt and compare are:

1. a general-to-specific strategy whereby we start pruning the coefficients that appear to be statistically insignificant (using Bollerslev and Wooldridge robust standard errors) in the most general expressions and go on to search down to the level where all coefficients are significant and
2. the smallest value of the Schwartz information criteria (BIC) among the 64 models.

The chosen models from the general-to-specific selection procedure are reported in Table 2.

The results show that, when evaluated in terms of coefficient significance, the inclusion of other variables in the information set appears to add explanatory power in each of the expressions. The model for absolute returns includes an asymmetric effect captured by  $r_{t-1}$  (the term  $r_{t-1}^2 d_{t-1}$  seems to be less important), and both the squared daily range and the squared daily volatility. The model for the high–low range seems to be the most parsimonious with the presence of just an asymmetric response of  $h_t^h$  to lagged values of the returns. Interestingly, the one model which attracts the highest number of significant variables is the model for the daily realized volatility in which there appear to be asymmetric effects from all variables, as well as lagged squared returns and lagged squared daily range. Some diagnostics (reported in the top panel of Table 3) show that there are no major specification problems: for reference we give the values of the BIC and the estimated log-likelihood, as well as the results of an ARCH(2) test (5% critical value = 5.99) and the Ljung–Box test  $Q(12)$  for the squared residuals (5% critical value = 21.03).

The specification search guided by the lowest value of the BIC gives the results we present in Table 4. The model selected for the daily range is the same as before. For absolute returns, the model selected here does not contain lagged square daily volatility; the values of the coefficients do not change much, so the profile of forecasts between the two models will differ mainly because of the impact of this variable. The model for the daily volatility here does not include the terms involving the daily range, and the values of the coefficients are fairly different. Again (bottom panel of Table 3) no major problems are signaled by the residual diagnostics.

#### 4. Model performance

The three models thus estimated could be used separately for one-step-ahead predictions using the estimated coefficients and the actual value of the right-hand

Table 2  
S&P500—General-to-specific model selection

$$\begin{aligned}
 h_t^r &= 3.389 - 0.031 r_{t-1}^2 + 0.901 h_{t-1}^r - 0.747 r_{t-1} + 0.11 h_{t-1}^2 - 0.010 v_{t-1}^2 \\
 &\quad \begin{matrix} 2.123 & -1.121 & 45.537 & -3.552 & 2.535 & 2.700 \end{matrix} \\
 h_t^h &= 7.622 + 0.109 h_{t-1}^2 + 0.850 h_{t-1}^h - 0.878 r_{t-1} \\
 &\quad \begin{matrix} 4.885 & 5.407 & 32.713 & -3.608 \end{matrix} \\
 h_t^v &= 1.654 - 0.021 v_{t-1}^2 + 0.728 h_{t-1}^v - 1.596 r_{t-1} + 0.211 v_{t-1} d_{t-1} + 0.094 r_{t-1}^2 + 0.098 h_{t-1}^2 + 0.114 h_{t-1}^2 d_{t-1} \\
 &\quad \begin{matrix} 4.005 & 1.292 & 80.909 & -14.932 & 7.278 & 9.801 & 4.286 & 4.127 \end{matrix}
 \end{aligned}$$

Equations for the time-varying component in the equations for the squares of absolute returns, high–low range, and daily realized volatility. Sample period January 4, 1988–December 30, 1997 (robust *t*-statistics in small font under the parameter value).

Table 3  
S&P500—diagnostics on selected models

	BIC	ARCH(2)	Q(12)	LOGLIK
<i>General-to-specific model selection</i>				
Absolute returns	7.8693	1.770	4.016	−9868.28
Daily range	7.8622	2.25	5.86	−9867.127
Realized volatility	7.2581	0.425	15.248	−9092.116
<i>Smallest BIC model selection</i>				
Absolute returns	7.8682	1.272	4.174	−9870.81
Daily range	7.8622	2.25	5.86	−9867.127
Realized volatility	7.2561	0.470	12.136	−9097.375

Sample period January 4, 1988–December 30, 1997.

Table 4  
S&P500—smallest BIC model selection

$$\begin{aligned}
 h_t^r &= 5.026 - 0.030 r_{t-1}^2 + 0.901 h_{t-1}^r - 0.745 r_{t-1} + 0.101 h_{t-1}^2 \\
 &\quad \text{2.805} \quad \text{1.068} \quad \text{43.432} \quad \text{3.293} \quad \text{2.328} \\
 h_t^h &= 7.622 + 0.109 h_{t-1}^2 + 0.850 h_{t-1}^h - 0.878 r_{t-1} \\
 &\quad \text{4.885} \quad \text{5.407} \quad \text{32.713} \quad \text{3.608} \\
 h_t^v &= 2.123 + 0.035 v_{t-1}^2 + 0.736 h_{t-1}^v - 1.183 r_{t-1} + 0.122 v_{t-1} d_{t-1} + 0.123 r_{t-1}^2 \\
 &\quad \text{8.061} \quad \text{2.366} \quad \text{91.479} \quad \text{28.350} \quad \text{6.688} \quad \text{23.911}
 \end{aligned}$$

Equations for the square of the time-varying component in absolute returns, high–low range, and daily realized volatility. Sample period January 4, 1988–December 30, 1997 (robust *t*-statistics in small font under the parameter value).

side variables: the performance of these single-equation models could be individually evaluated in relationship to the performance of the corresponding GARCH models. The interest of what is being done here, though, lies in the fact that the three equations together can be seen as a system which can be used as a tool for multi-step forecasting for medium horizons. Let us consider the left-hand side as a three dimensional vector  $h_t$ , and consider that at time  $T + 1$  we have

$$h_{T+1|T} = \begin{pmatrix} h_{T+1|T}^r \\ h_{T+1|T}^h \\ h_{T+1|T}^v \end{pmatrix} = \begin{pmatrix} \omega_r \\ \omega_h \\ \omega_v \end{pmatrix} + \mathbf{A}^*(r_T, r_T^2, h_T^2, h_T^2 d_T, v_T^2, v_T^2 d_T, h_T^r, h_T^h, h_T^v)', \tag{18}$$

where  $\mathbf{A}^*$  is a  $3 \times 9$  matrix which includes the coefficients on the variables the value of which is known at time  $T$ . To forecast the various future second-order moments conditional on information at time  $T$  for maturities greater than 1, we need to substitute the right-hand side variables with their conditional expectation as of time

T. For a generic horizon  $k$ , we will have

$$\begin{aligned}
 E_T(r_{T+k-1}) &= 0, \\
 E_T(r_{T+k-1}^2) &= h_{T+k-1|T}^r, \\
 E_T(h_{T+k-1}^2) &= h_{T+k-1|T}^h, \\
 E_T(h_{T+k-1}^2 d_{T+k-1}) &= \frac{1}{2} h_{T+k-1|T}^h, \\
 E_T(v_{T+k-1}^2) &= h_{T+k-1|T}^v, \\
 E_T(v_{T+k-1}^2 d_{T+k-1}) &= \frac{1}{2} h_{T+k-1|T}^v
 \end{aligned}$$

and, therefore expression (18) is substituted by

$$h_{T+k|T} = \begin{pmatrix} h_{T+k|T}^r \\ h_{T+k|T}^h \\ h_{T+k|T}^v \end{pmatrix} = \begin{pmatrix} \omega_r \\ \omega_h \\ \omega_v \end{pmatrix} + \mathbf{A} \begin{pmatrix} h_{T+k-1|T}^r \\ h_{T+k-1|T}^h \\ h_{T+k-1|T}^v \end{pmatrix} = \omega + \mathbf{A}h_{T+k-1|T}. \tag{19}$$

The dynamic properties of the estimated system can therefore be evaluated by examining the characteristic roots of the matrix  $\mathbf{A}$ . In principle, there could be stable complex conjugate roots that would give the system some dampened cyclicity in the forecasting. For the case at hand the roots for the two sets of estimates are given as in Table 5.

The roots are fairly similar across the two selection procedures: one can therefore expect that the two multi-equation models will provide different forecasting profiles for the short horizon, while they will tend to be very similar in the medium to long run. For this reason we will report the graphs just for the models selected according to the smallest BIC criterion.

We will evaluate the performance of the models by first considering the profile of the out-of-sample forecasts generated by the system. Let us start on the first day after the estimation sample (January 2, 1998), and let us take the observations recorded on December 30, 1997 as the starting values for the dynamic forecasts 132 periods ahead (hence from January 2, 1998 to July 13, 1998). Then let us move the starting date five days ahead (i.e. January 8, 1998), and collect the forecasts for the same 132 days horizon. We will keep on moving ahead by five days and we will repeat the procedure until the last considered starting value corresponds to March 6, 1998 (forecasting horizon from March 9, 1998 to September 14, 1998). What we obtain are different profiles which can be superimposed as in Fig. 4: the different lines start progressively

Table 5  
Characteristic roots of the matrix  $\mathbf{A}$

Model selection			
General-to-specific	0.958	0.860	0.833
Smallest BIC	0.958	0.870	0.832

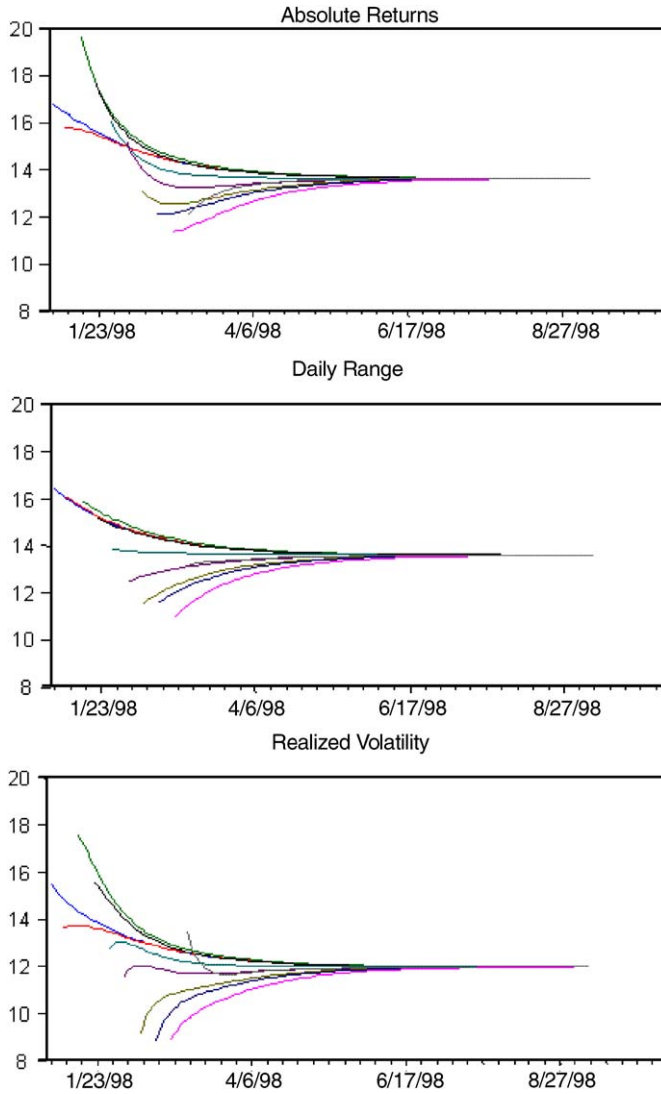


Fig. 4. The S&P500 index: out-of-sample volatility forecasts: abs. returns, high–low, daily volatility. Various starting dates.

at later and later periods, they last 132 periods and converge to the same value (unconditional variance). The forecasts for the daily range indicator exhibit the typical monotonic profile of a GARCH(1,1) model, since the model is the closest to the base specification. This is not the case for the curves corresponding to forecasts obtained from the other two indicators: it is interesting to note that the forecasts produced by our model may have a non-monotonic behavior with over- (or under-) shooting of their long term (unconditional) values at intermediate horizons.



Let us now consider the volatility for an asset (or an index) at a given maturity  $T+k$  as the square root of the cumulated sum of  $j$ -step-ahead forecasts generated by expression (18), when  $j = 1$ , or (19), when  $j$  is between 2 and  $k$  (cf. Engle and Patton, 2001). This corresponds to evaluating the square root of the cumulated sum of the expected values of each volatility indicator at any time between 1 and  $k$ . As  $k$  increases the terms of the sum will tend to repeat themselves. We thus have

$$\begin{pmatrix} v_{T+k|T}^a \\ v_{T+k|T}^h \\ v_{T+k|T}^v \end{pmatrix} = \begin{pmatrix} \sqrt{\sum_{j=1}^k h_{T+j|T}^r} \\ \sqrt{\sum_{j=1}^k h_{T+j|T}^h} \\ \sqrt{\sum_{j=1}^k h_{T+j|T}^v} \end{pmatrix} = \begin{pmatrix} \sqrt{\sum_{j=1}^k E_T r_{T+j}^2} \\ \sqrt{\sum_{j=1}^k E_T h_{T+j}^2} \\ \sqrt{\sum_{j=1}^k E_T v_{T+j}^2} \end{pmatrix}. \tag{20}$$

For reasons that will be clearer in the next section when we discuss the comparison of these forecasts with the VIX volatility index, we choose a horizon  $k$  equal to 22, that is, a one-month ahead forecasts. In Fig. 5 we report the values of the cumulative volatility forecast as the (square root of the) average of 1-step, 2-steps, . . . , 22-steps ahead out-of-sample forecasts obtained by the general-to-specific three-equation system (18) and (19) relative to the standard GARCH(1,1) specification. As one would expect, for the equation for the range indicator, the values obtained with our model and with the standard base specification are quite similar since the system

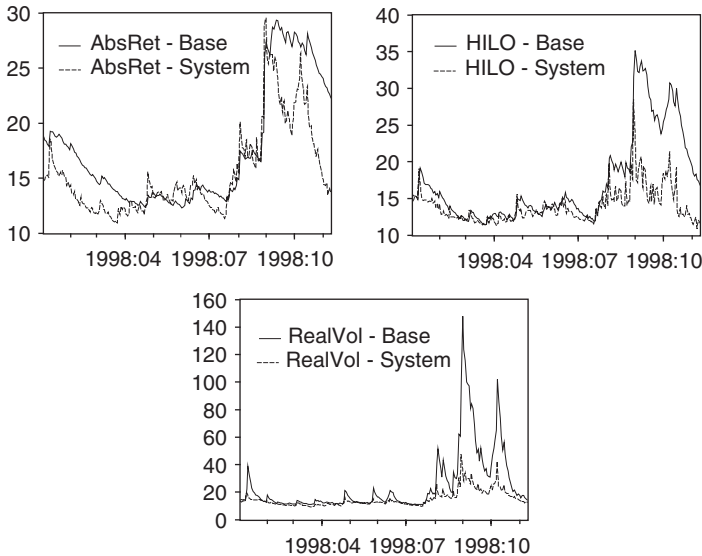


Fig. 5. The S&P500 index: one month-ahead forecasts of volatility: abs. returns, high–low, realized volatility. Out of sample—January 2, 1998–November 10, 1998.

specification contains just lagged returns on top of the base specification. The results show that the estimates obtained with the system specification are generally higher and more persistent than the estimates obtained with the base specification in the case of the absolute returns whereas the reverse is true for the daily realized volatility results. A remarkable difference is observed for estimates on or about August 31, 1998 for which the base specification provides much higher estimates. Whether this signals excessive volatility forecasts by the latter set of models is an issue that requires a more specific type of evaluation. What we need is an overall comparison of the two sets of forecasts gauged in reference to a market-based volatility measure such as the volatility index VIX. We turn to this in the next section.

## 5. Model based and market based volatility

The Chicago Board of Options Exchange (CBOE) is the world's largest options exchange where standardized stock and index options are traded.<sup>5</sup> Starting in 1993, CBOE has computed a volatility index called VIX with the aim of measuring market expectations of short term volatility as implicit in stock index option prices.<sup>6</sup> Originally computed as the weighted average of the implied volatilities from eight at-the-money call and put options on the S&P100 index (OEX) which have an average time to maturity of 30 days, in 2003 the VIX has been entirely revised by changing the reference index to the S&P500, taking into account a wide range of strike prices for the same 30 day maturity, and freeing its calculation from any specific option pricing model.<sup>7</sup> The behavior of the series from January 2, 1990 to December 14, 1998 is shown in Fig. 6. Standard Augmented Dickey Fuller tests show that the unit root hypothesis is rejected, although the degree of persistence in the series is very high.

For the purposes of this paper, the analysis of VIX in reference to volatility estimates with a multiple indicator conditional model is relevant because we can use the value of VIX as a reference for our volatility forecasts. Under several auxiliary assumptions, the optimal forecast of volatility from past prices should match the Black–Scholes at-the-money implied volatility; under even weaker conditions, the VIX should also represent future volatility. However, information such as forthcoming announcements known to traders but not to the econometricians may lead to episodes of positive or negative discrepancies. The 22-period-ahead horizon for the term structure of volatilities was chosen to ensure compatibility with the 30 calendar day horizon considered in the construction of the VIX.

<sup>5</sup>For more detailed information, cf. the CBOE web site [www.cboe.com](http://www.cboe.com).

<sup>6</sup>The value of the index is reckoned to measure investors' fears. A very high value signals bearishness as downward risk is perceived to be higher than upward risk. Chartists look at VIX for possible trend reversals.

<sup>7</sup>For more details, cf. [www.cboe.com/micro/vix/introduction.aspx](http://www.cboe.com/micro/vix/introduction.aspx). VIX covers the period from January 2, 1990 to present while data for the old VIX are available from January 2, 1986 to present, supplied under the name VXO.

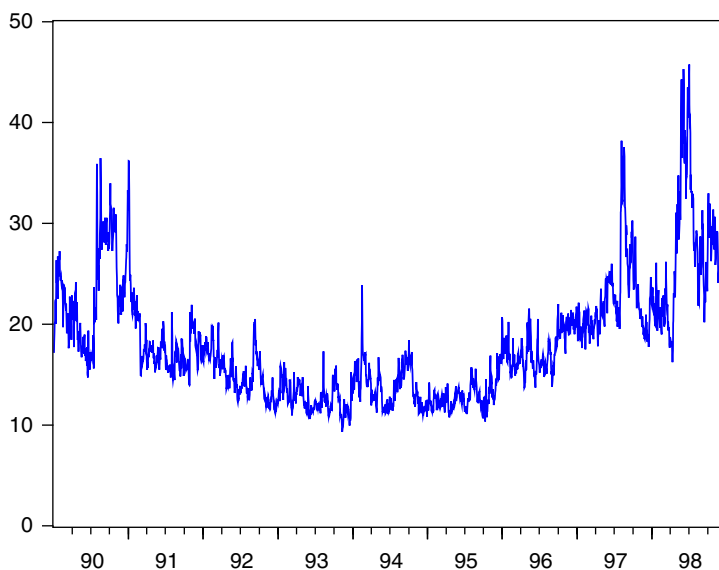


Fig. 6. The CBOE VIX index: January 2, 1990–December 14, 1998.

We will therefore examine two main issues related to the incremental explanatory value of the forecasts obtained with the base and with the system specifications for each indicator:

- looking at them by indicator, we will test whether the coefficients of the forecasts are pairwise equal to zero in a regression of VIX on an AR(1) term and on the volatility forecasts, both in and out of sample;
- looking at them by specification, we will test whether the three coefficients of the forecasts of the base specification and the three coefficients of the system specification are jointly equal to zero in the same regressions.

In order to answer these questions we ran two sets of regressions each over two separate periods (estimation sample and out-of-sample). We use both the forecasts provided by the models selected according to the smallest BIC criterion and to general-to-specific model selection procedure, since they provide partially different outcomes. Two models were estimated using VIX as the dependent variable: one is a simple autoregressive representation and the second adds the one month-ahead volatility forecasts according to the base and the system specifications for the MEMs as independent variables. It is not surprising that there would be serial correlation in the estimated residuals. Differences between the optimal econometric forecast of volatility and implied volatilities would naturally arise from the reduced information set used by the econometrician. For example, an upcoming election would be incorporated in VIX but not GARCH and this would persist for many days. Yet, it is of interest whether the model based forecasts can provide some incremental explanatory enhancement in the evolution of the volatility index.

The results are shown in Table 6 where we report the results of the in-sample and out-of-sample analysis with the top panel referring to the smallest BIC selection criterion and the bottom one to the results obtained with the general-to specific approach. The results show that, with a few exceptions, all coefficients are individually statistically significant. One notices that in most specifications the constant term is highly significant: when this parameter is positive, the volatility estimates would confirm a recorded feature of the volatility index to underestimate the implied volatility level (it could be due to a volatility risk premium). The contribution of various sets of forecasts is assessed by means of HAC Wald *F*-tests and reported in the lower part of each panel. We test whether each forecast has an impact by type of indicator (rows labeled absolute returns, daily range or realized volatility) or by type of method (base specification or system specification). There is no clear indication of a better in-sample model performance as a consequence of the selection criterion adopted. In both sets of results one notices that negative coefficients associated with the daily range forecasts appear: concentrating on the high–low range, for example, other things being equal, an increase in the forecast of the volatility value for the range is associated with a reduction in the value of VIX.

Table 6  
Regression results for VIX with and without the 22-day-ahead forecasts of volatility obtained using the base and the system specifications

		In sample		Out of sample	
<i>Lowest BIC</i>					
	Constant	16.997** (19.61)	−2.660* (−2.33)	26.121** (7.60)	11.093* (2.24)
	AR(1)	0.976** (148.38)	0.915** (57.45)	0.967** (53.20)	0.974** (48.49)
Multi-step average volatility (base specification)	Absolute return		1.037** (7.17)		0.364 (0.95)
	Daily range		−1.400** (−8.114)		−2.314** (−5.40)
	Realized volatility		−0.049** (−2.83)		−0.069 (−1.52)
Multi-step average volatility (system specification)	Absolute return		3.614** (7.91)		4.403** (11.09)
	Daily range		−1.865** (−3.64)		−1.555** (−5.88)
	Realized volatility		0.261** (3.42)		0.290* (2.37)
Diagnostics	<i>R</i> -squared	0.9512	0.9714	0.9348	0.9777
	TR <sup>2</sup> for AR(4)	25.089**	28.227**	13.540**	15.041**
	TR <sup>2</sup> for ARCH(4)	150.63**	241.64**	44.929**	19.562**

Table 6 (continued)

		In sample		Out of sample	
Robust Wald tests of joint zero coefficients	Abs. returns terms		65.040**		70.919**
	Daily range terms		50.989**		26.027**
	Realized volatility terms		18.320**		2.802
	Base terms		43.382**		30.855**
	System terms		113.25**		63.681**
<i>General to specific</i>					
	Constant	16.997** (19.61)	−0.397 (−0.37)	26.121** (7.60)	16.511* (1.92)
	AR(1)	0.976** (148.38)	0.916** (57.98)	0.967** (53.20)	0.981** (40.58)
	Multi-step average volatility (base specification)		1.186** (8.62)		0.523 (1.21)
	Daily range		−1.585** (−9.59)		−2.025** (−3.85)
	Realized volatility		0.065** (3.07)		0.132** (3.55)
	Multi-step average volatility (system specification)		3.030** (8.32)		3.105** (5.84)
	Daily range		−1.731** (−3.86)		−1.580** (−4.88)
	Realized volatility		0.484** (7.15)		0.587** (9.91)
	Diagnostics	<i>R</i> -squared	0.9512	0.9713	0.9348
	TR <sup>2</sup> for AR(4)	25.089**	25.133**	13.540**	4.094*
	TR <sup>2</sup> for ARCH(4)	150.63**	218.512**	44.929**	10.916**
	Robust Wald tests of joint zero coefficients	Abs. returns terms		63.288**	
	Daily range terms		57.933**		11.954**
	Realized volatility terms		27.279**		55.295**
	Base terms		33.489**		15.662**
	System terms		105.97**		53.551**

Robust *t*-values under the estimated coefficient values.

\* = significant at 5%.

\*\* = significant at 1%.

This could be a sign of the daily range values being indicators of a trend reversal in periods of high volatility, but the issue deserves further attention.

The same kind of analysis may be carried over to an out-of-sample combination of forecasts exercised over the period January 2, 1998–December 14, 1998, in which the same variable (VIX) is regressed on a constant and an AR(1) term and then the

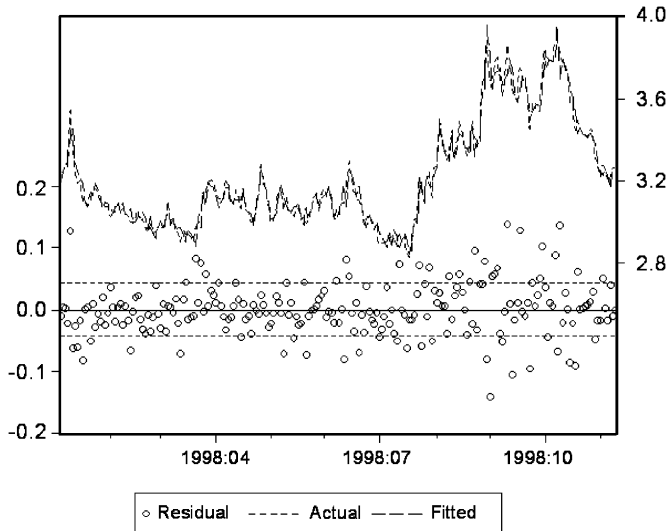


Fig. 7. The CBOE VIX index: out of sample combination of forecasts January 2, 1998–November 10, 1998—AR(1) model with MEM forecasts (smallest BIC—coefficients in Table 6, last column).

22-day-ahead volatility forecasts obtained from the same models (keeping the coefficients fixed at their estimated in-sample values) are added as regressors. The results, again broken down between lowest BIC models and general-to-specific models, are presented in the last columns under the heading out of sample. Once again, the significance of each set of forecasts is maintained from what the *F*-tests are signaling. The lowest BIC provides a slightly better performance: all parameters are significant. Some features of this combination of forecasts can also be assessed graphically (Fig. 7) since it seems that the highest variance appears to correspond to periods in which the VIX index is higher and more volatile.

### 6. Conclusions

The motivation for this paper stems from the consideration that various volatility indicators used in the literature (absolute daily returns, daily high–low range and intra-daily realized volatility) may present features which should be jointly combined in a dynamic model to enhance the information content of individual measures. We have adopted a novel approach, called multiplicative error model (Engle, 2002), which is suited to model the conditional behavior of positively valued variables choosing a convenient GARCH-type structure to model persistence. It turns out that a wide variety of error assumptions can be accommodated with a standard easy-to-use procedure and inference conducted on the basis of the robust variance covariance matrix. For the problem at hand the chosen specification for the MEM is multivariate and is suitable to be dynamically solved for short to medium range

forecasting horizons. For the data at hand (Standard and Poor 500), we show that the approach is rewarding in that the retained specifications for each indicator are augmented by the presence of lagged values of other indicators. In particular, we obtain the result that daily range and returns have explanatory power for realized volatility, and daily range is the indicator with the most parsimonious model.

We evaluate the performance of this model by producing 22-step-ahead volatility forecasts for each of the three indicators and by using them in a regression framework to detect their explanatory power for an index of market volatility such as the VIX index. The results show that the model performs well, with significant persistence of the VIX index being captured by this forecasts both individually and grouped either by type of indicator and by type of specification adopted. We thus provide evidence that model based forecasts have significant explanatory power in tracking the value of a market based volatility measure.

The estimator thus derived is efficient when the choice of error distribution is within the class of a Gamma distribution: it retains consistency when other more flexible and time-varying density specifications may be more appropriate and research is under way to identify such densities. By the same token, in the present context we adopted a diagonal structure for the variance covariance matrix of the error terms, and further efficiency gains could be achieved by considering more complex structures of correlations among the error terms, paralleling the literature on multivariate GARCH models.

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## References

- Alizadeh, S., Brandt, M.W., Diebold, F.X., 2002. Range based estimation of stochastic volatility models. *Journal of Finance* 57, 1047–1091.
- Andersen, T.G., Bollerslev, T., 1997. Heterogeneous information arrivals and return volatility dynamics: uncovering the long-run in high frequency returns. *Journal of Finance* 52, 975–1005.
- Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2000. Great realizations. *Risk* 13, 105–108.
- Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2001. The distribution of realized exchange rate volatility. *Journal of the American Statistical Association* 96, 42–55.
- Andersen, T.G., Bollerslev, T., Diebold, F.X., 2002. Parametric and nonparametric volatility measurement. In: Ait-Sahalia, Y., Hansen, L.P. (Eds.), *Handbook of Financial Econometrics*. North-Holland, Amsterdam, forthcoming.

- Baillie, R.T., Bollerslev, T., Mikkelsen, H.O., 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 74, 3–30.
- Bardorff-Nielsen, O., Shephard, N., 2002. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society, Series B* 64, 253–280.
- Beckers, S., 1983. Variances of security price returns based on high, low, and closing prices. *Journal of Business* 56, 97–112.
- Blair, B.J., Poon, S.-H., Taylor, S.J., 2001. Forecasting S&P 100 volatility: the incremental information content of implied volatilities and high-frequency index returns. *Journal of Econometrics* 105, 5–26.
- Bollerslev, T., Mikkelsen, H.O., 1996. Modeling and pricing long memory in stock market volatility. *Journal of Econometrics* 73, 151–184.
- Bollerslev, T., Wooldridge, J.M., 1992. Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances. *Econometric Reviews* 11, 143–172.
- Brandt, M.W., Diebold, F.X., 2003. A no-arbitrage approach to range-based estimation of return covariances and correlations. Working Paper No. 03-013, Penn Institute for Economic Research.
- Brandt, M.W., Jones, C.S., 2002. Volatility forecasting with range-based EGARCH models. Mimeo, University of Pennsylvania.
- Breidt, F.J., Crato, N., de Lima, P., 1998. The detection and estimation of long memory in stochastic volatility. *Journal of Econometrics* 83, 325–348.
- Brunetti, C., Lildholdt, P., 2003. Return-based and range-based (co)variance estimation—with an application to foreign exchange markets. Mimeo, University of Pennsylvania.
- Chou, R., 2001. Forecasting financial volatilities with extreme values: the conditional autoregressive range (CARR) model. Mimeo, Academia Sinica.
- Deo, R.S., Hurvich, C.M., 1999. Estimation of long memory in volatility. Mimeo, Stern Business School.
- Ding, Z., Granger, C.W.J., Engle, R.F., 1993. A long memory property of stock market returns and a new model. *Journal of Empirical Finance* 1, 83–106.
- Engle, R.F., 2000. The econometrics of ultra high frequency data. *Econometrica* 68, 1–22.
- Engle, R.F., 2002. New frontiers for ARCH models. *Journal of Applied Econometrics* 17, 425–446.
- Engle, R.F., Lee, G.J., 1999. A permanent and transitory component model of stock return volatility. In: Engle, R.F., White, H. (Eds.), *Cointegration, Causality, and Forecasting: A Festschrift in Honor of C.W.J. Granger*. Oxford University Press, Oxford, pp. 475–497.
- Engle, R.F., Patton, A., 2001. What good is a volatility model? *Quantitative Finance* 1, 237–245.
- Engle, R.F., Russell, J.R., 1998. Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica* 66, 1127–1162.
- Gallant, A.R., Hsu, C.-T., Tauchen, G., 1999. Using daily range data to calibrate volatility diffusions and extract the forward integrated variance. *Review of Economics and Statistics* 81, 617–631.
- Garman, M.B., Klass, M.J., 1980. On the estimation of security price volatilities from historical data. *Journal of Business* 53, 67–78.
- Ghose, D., Kroner, K., 1996. Components of volatility in foreign exchange markets: an empirical analysis of high frequency data. Mimeo, Department of Economics, University of Arizona.
- Granger, C.W.J., Sin, C.-Y., 2000. Modelling the absolute returns of different stock indices: exploring the forecastability of an alternative measure of risk. *Journal of Forecasting* 19, 277–298.
- Lee, S.-W., Hansen, B.E., 1994. Asymptotic theory for the GARCH(1,1) quasi-maximum likelihood estimator. *Econometric Theory* 10, 29–52.
- Manganelli, S., 2002. Duration, volume, and volatility impact of trades. Working Paper No. 125, European Central Bank.
- Oomen, R.C.A., 2002. Modelling realized variance when returns are serially correlated. Mimeo, Warwick Business School.
- Parkinson, M., 1980. The extreme value method for estimating the variance of the rate of return. *Journal of Business* 53, 61–66.
- Rogers, L.C.G., Satchell, S.E., 1991. Estimating variance from high, low and closing process. *Annals of Applied Probability* 1, 504–512.
- Roll, R., 1984. A simple implicit measure of the effective bid-ask spread in an efficient market. *Journal of Finance* 39, 1127–1139.



- Schwert, G.W., 1998. Stock market volatility: ten years after the crash. *Brookings–Wharton Papers on Financial Services*. Brookings Institutions, Washington, DC.
- Taylor, S.J., 1986. *Modelling Financial Time Series*. Wiley, Chichester.
- Taylor, S.J., 1987. Forecasting the volatility of currency exchange rates. *International Journal of Forecasting* 3, 159–170.
- Zhou, B., 1996. High-frequency data and volatility in foreign-exchange rates. *Journal of Business and Economic Statistics* 14, 45–52.