Job Shop Problem

Job shop problem $J||C_{\max}$ is NP-hard if either $m\geq3$ or $n\geq3$.

We first consider the special with m=2 machines (Section 1) and then discuss the branch and bound algorithm for an arbitrary number of machines (Sections 2 and 3).

1. Exact algorithm for $J2|\,|C_{\max}$

Consider a job shop problem $J2||C_{\max}$ with m=2 machines in which each job has at most two operations. This problem can be solved by a reduction to the two-machine flow shop problem $F2||C_{\max}$.

The set of jobs $N=\{1,\ldots,n\}$ can be split into two subsets:

N_{AB} - the jobs that consist of two operations with the processing route (A, B);

N_{BA} - the jobs that consist of two operations with the processing route (B, A).

The algorithm presented below is due to R. Jackson and was first described in 1956. It uses Johnson's flow shop algorithm as a subroutine.

<u>Jackson's Algorithm</u>

1. Run Johnson's algorithm with the set of jobs N_{AB} and find the corresponding sequence R_{AB}.
2. Run Johnson's algorithm with the set of jobs N_{BA} and find the corresponding sequence R_{BA}.
3. On machine A, first schedule N_{AB} according to R_{AB} and then N_{BA} according to R_{BA}.
4. On machine B, first schedule N_{BA} according to R_{BA} and then N_{AB} according to R_{AB}.

<u>Example</u>. Eleven patients came to a surgery run by two doctors, A and B. Let {1,2,3,4} be the set of patients who want to see doctor A before doctor B; the patients of set {5,6,7} want to see doctor B before doctor A; the patients of set {8,9} want to see only doctor A; the patients of set {10,11} want to see only doctor B. The times for each visit (in suitable time units) are given in the table:

j	1	2	3	4	5	6	7	8	9	10	11
a_j	3	2	1	1	2	4	3	1	2		
b_j	2	1	2	1	4	8	9			2	1

Find a schedule with the smallest makespan.

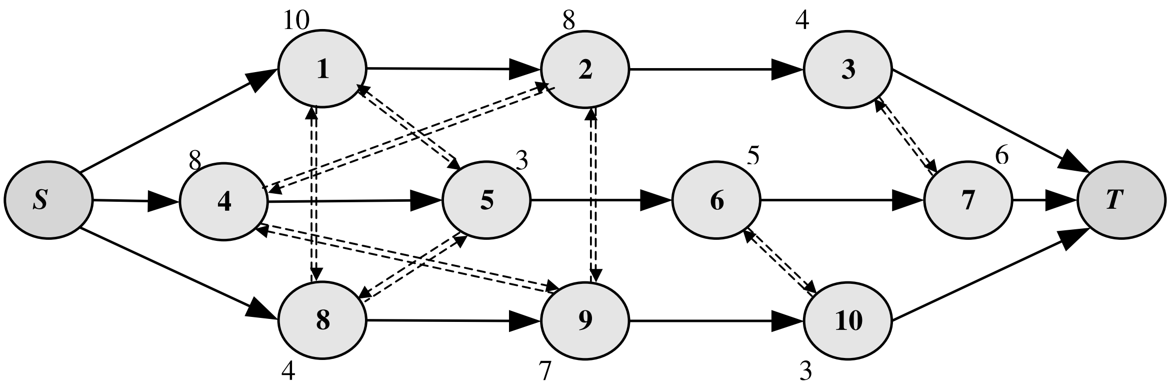
2. Disjunctive graph model for the job shop problem with *m* machines

In a disjunctive graph model, each operation of a job is represented by a node. Additionally there are two dummy nodes *S* (starting vertex) and *T* (terminating vertex). All operations of one job are connected by conjunctive (oriented) arcs; all operations processed by the same machine are connected by disjunctive arcs.

Example:
Consider a job shop problem with n=3 jobs and m=4 machines:

Jobs	Machine Sequence	Processing times
1	1, 2, 3	p_{11}=10, p_{21}=8, p_{31}=4
2	2, 1, 4, 3	p_{12}=3, p_{22}=8, p_{32}=6, p_{42}=5
3	1, 2, 4	p_{13}=4, p_{23}=7, p_{43}=3

Disjunctive graph is illustrated in the following figure.



For simplicity, we renumber all operations as 1, 2, …, 10.

Our objective is to find the order of the operations on each machine, i.e., to fix one arc from each pair of disjunctive arcs.

3. Branch and Bound Algorithm

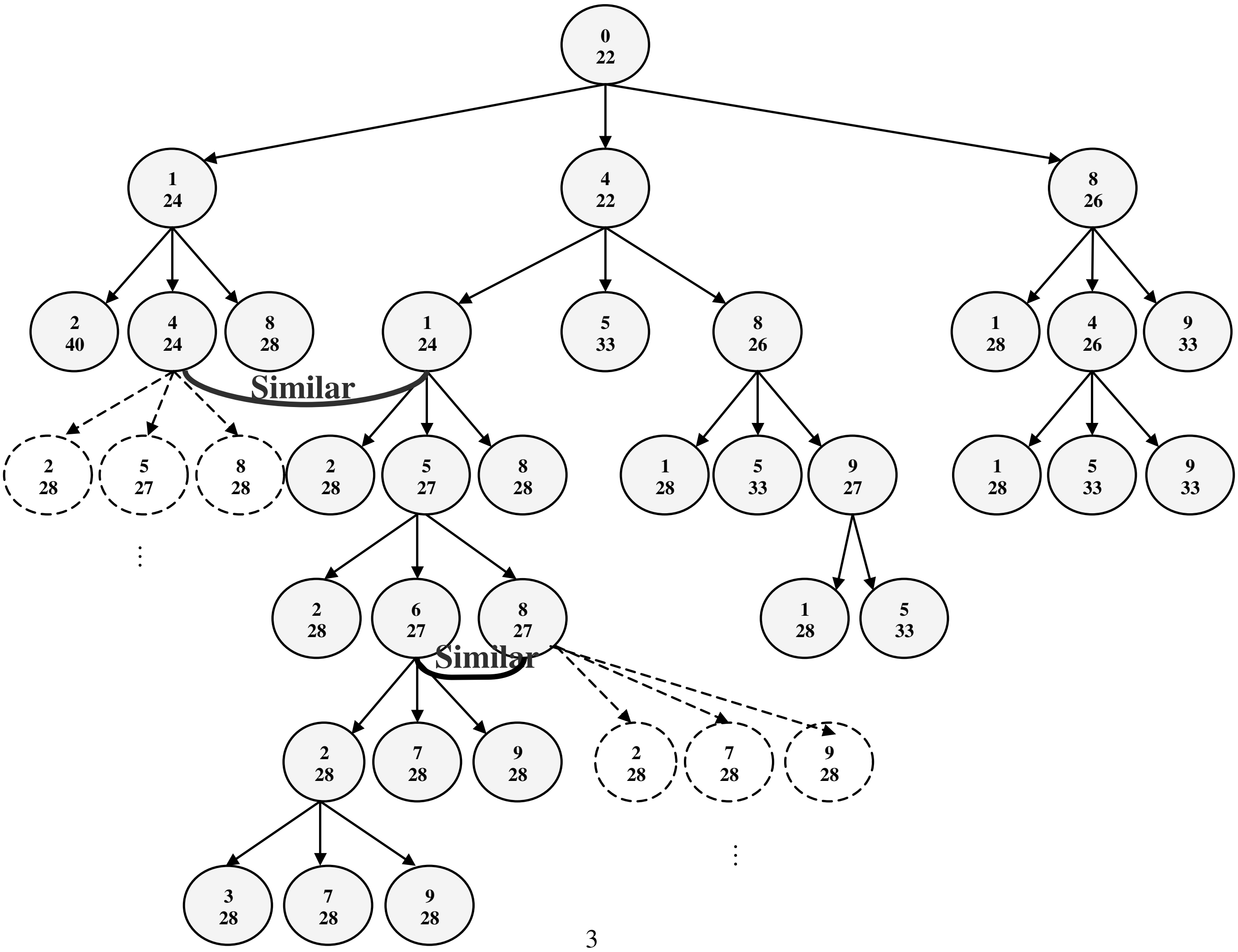
Let O denote the set of “schedulable operations” (whose predecessors have already been scheduled). Initially O contains the first operation of each job.

Step 1: For each operation *i*, calculate its starting time as the length of the longest path from the starting vertex *S* to *i* (ignore disjunctive arcs). Determine the Lower Bound of the makespan (*LB*) as the starting time of the dummy operation *T*.

Step 2: *Schedule* successively each operation in O as the next one on its machine.
“*Scheduling*” an operation (say, operation *i*) means replacing each pair of disjunctive arcs incident to *i* by a conjunctive arc starting from *i*.
For each such scheduling,
— delete operation *i* from O;
— include immediate followers in O;
— replace corresponding pairs of disjunctive arcs by conjunctive arcs;
— recalculate the starting times of all operations.

Repeat Step 2 for each resulting graph until all operations are scheduled.

The search tree is represented in the following figure.



In the figure, for each node of the tree we specify the operation, which is being scheduled, and the lower bound of the makespan.

4. Conclusions

Job shop problem remains one of the most difficult combinatorial problems to date, and always arouses new research interest.

In 1963 Fischer and Thompson posed a small example of the jobs shop problem with 10 jobs and 10 machines. The example remained open for 25 years. It was finally solved by
- Carlier & Pinson (1988),
- Applegate & Cook (1991),
- Brucker, Jurisch & Sievers (1994).

[1] H. Fisher and G.L. Thompson (1963) Probabilistic learning combinations of local job-shop scheduling rules, in: J.F. Muth and G.L. Thompson (eds.), *Industrial Scheduling*, Prentice-Hall, Englewood Cliffs, pp.225 – 251.

[2] J. Carlier and E. Pinson (1988) An algorithm for solving job shop problem. *Management Science* 35, 164 – 176.

[3] D. Applegate and W. Cook (1991) A computational study of the job-shop scheduling problem. *ORSA Journal of Computing* 3, 149 – 156.

[4] P. Brucker, B. Jurisch and B. Sievers (1994) A branch and bound algorithm for the job shop scheduling problem. *Discrete Applied Mathematics* 49, 107 – 127.