Single machine models: Number of Tardy Jobs

Problem $1||\sum U_j$:

- Structure of an optimal schedule:
  - set S_1 of jobs meeting their due dates
  - set S_2 of jobs being late
  - jobs of S_1 are scheduled before jobs from S_2
  - jobs from S_1 are scheduled in EDD order
  - jobs from S_2 are scheduled in an arbitrary order
- Result: a partition of the set of jobs into sets S_1 and S_2 is sufficient to describe a solution

Single machine models: Number of Tardy Jobs

Algorithm $1||\sum U_j$

1. enumerate jobs such that $d_1 \leq \ldots \leq d_n$;
2. $S_1 := \emptyset;\ t := 0$;
3. FOR j:=1 TO n DO
4. $\quad S_1 := S_1 \cup \{j\}; t := t + p_j$;
5. $\quad$ IF $t > d_j$ THEN
6. $\qquad$ Find job k with largest p_k value in S_1;
7. $\qquad S_1 := S_1 \setminus \{k\}; t := t - p_k$;
8. $\quad$ END
9. END

Single machine models: Number of Tardy Jobs

Remarks Algorithm $1||\sum U_j$

- Principle: schedule jobs in order of increasing due dates and always when a job gets late, remove the job with largest processing time; all removed jobs are late
- complexity: $O(n\log(n))$
- Example: $n = 5$; $p = (7, 8, 4, 6, 6)$; $d = (9, 17, 18, 19, 21)$

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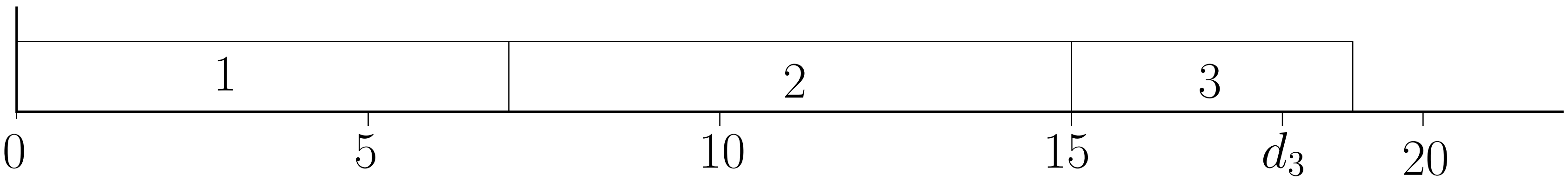

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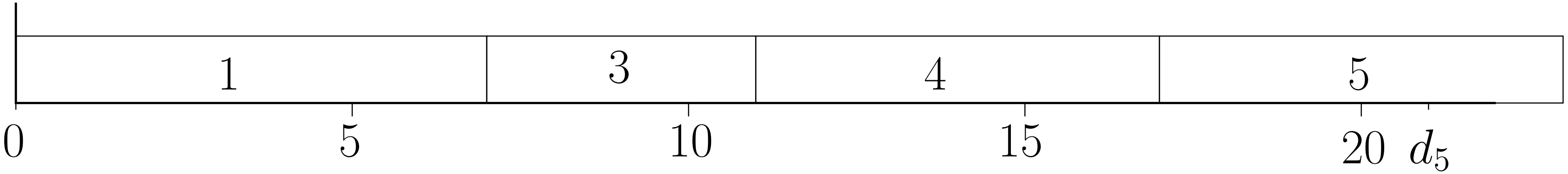

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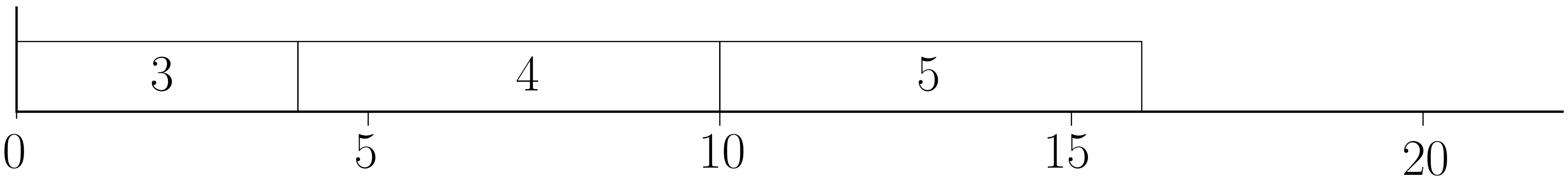

- Algorithm $1||\sum U_j$ computes an optimal solution
  Proof on the board

Single machine models: Weighted Number of Tardy Jobs

Problem $1||\sum w_j U_j$

- problem $1||\sum w_j U_j$ is NP-hard even if all due dates are the same; i.e. $1|d_j = d|\sum w_j U_j$ is NP-hard
  Proof on the board (reduction from PARTITION)
- priority based heuristic (WSPT-rule):
  schedule jobs in decreasing w_j/p_j order

Single machine models: Weighted Number of Tardy Jobs -1-

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- WSPT may perform arbitrary bad for $1||\sum w_jU_j$:

$n = 3; p = (1, 1, M);\ w = (1+\epsilon, 1, M-\epsilon); d = (1+M, 1+M, 1+M)$

$$\sum w_jU_j(WSPT)/\sum w_jU_j(opt) = (M-\epsilon)/(1+\epsilon)$$

Single machine models: Weighted Number of Tardy Jobs -2-

Dynamic Programming for $1||\sum w_jU_j$

- assume $d_1 \leq \ldots \leq d_n$
- as for $1||\sum U_j$ a solution is given by a partition of the set of jobs into sets S_1 and S_2 and jobs in S_1 are in EDD order
- Definition:
  - $F_j(t)$:= minimum criterion value for scheduling the first j jobs such that the processing time of the on-time jobs is at most t
- $F_n(T)$ with $T = \sum_{j=1}^{n} p_j$ is optimal value for problem $1||\sum w_jU_j$
- Initial conditions:

$$F_j(t) = \begin{cases} \infty & \text{for } t < 0;\ j = 1, \ldots, n \\ 0 & \text{for } t \geq 0;\ j = 0 \end{cases} \tag{1}$$

Single machine models: Weighted Number of Tardy Jobs

Dynamic Programming for $1||\sum w_jU_j$ (cont.)

- if $0 \leq t \leq d_j$ and j is late in the schedule corresponding to $F_j(t)$, we have $F_j(t) = F_{j-1}(t) + w_j$
- if $0 \leq t \leq d_j$ and j is on time in the schedule corresponding to $F_j(t)$, we have $F_j(t) = F_{j-1}(t - p_j)$

Single machine models: Weighted Number of Tardy Jobs -3-

Dynamic Programming for $1||\sum w_jU_j$ (cont.)

- if $0 \leq t \leq d_j$ and j is late in the schedule corresponding to $F_j(t)$, we have $F_j(t) = F_{j-1}(t) + w_j$
- if $0 \leq t \leq d_j$ and j is on time in the schedule corresponding to $F_j(t)$, we have $F_j(t) = F_{j-1}(t - p_j)$
- summarizing, we get for $j = 1, \ldots, n$:

$$F_j(t) = \begin{cases} \min\{F_{j-1}(t - p_j), F_{j-1}(t) + w_j\} & \text{for } 0 \leq t \leq d_j \\ F_j(d_j) & \text{for } d_j < t \leq T \end{cases} \tag{2}$$

Single machine models: Weighted Number of Tardy Jobs

<u>DP-algorithm for $1||\sum w_j U_j$</u>

1. initialize $F_j(t)$ according to (1)
2. FOR $j := 1$ TO n DO
3. FOR $t := 0$ TO T DO
4. update $F_j(t)$ according to (2)
5. $\sum w_j U_j(OPT) = F_n(d_n)$

Single machine models: Weighted Number of Tardy Jobs -4-

<u>DP-algorithm for $1||\sum w_j U_j$</u>

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4. update $F_j(t)$ according to (2)
5. $\sum w_j U_j(OPT) = F_n(d_n)$

- complexity is $O(n\sum_{j=1}^{n} p_j)$
- thus, algorithm is pseudopolynomial

Single machine models: Total Tardiness

Basic results:

- $1||\sum T_j$ is NP-hard
- preemption does not improve the criterion value
  $\rightarrow 1|pmtn|\sum T_j$ is NP-hard
- idle times do not improve the criterion value
- Lemma 1: If $p_j \leq p_k$ and $d_j \leq d_k$, then an optimal schedule exist in which job j is scheduled before job k.
  Proof: exercise
- this lemma gives a dominance rule

Single machine models: Total Tardiness -2-

Structural property for $1||\sum T_j$

- let k be a fixed job and $\hat{C}_k$ be latest possible completion time of job k in an optimal schedule
- define
$$\hat{d}_j = \begin{cases} d_j & \text{for } j \neq k \\ \max\{d_k, \hat{C}_k\} & \text{for } j = k \end{cases}$$
- Lemma 2: Any optimal sequence w.r.t. $\hat{d}_1, \ldots, \hat{d}_n$ is also optimal w.r.t. $d_1, \ldots, d_n$.
  Proof on the board

Single machine models: Total Tardiness

Structural property for $1||\sum T_j$ (cont.)

- let $d_1 \leq \ldots \leq d_n$
- let k be the job with $p_k = \max\{p_1, \ldots, p_n\}$
- Lemma 1 implies that an optimal schedule exists where

$$\{1, \ldots, k-1\} \rightarrow k$$

- Lemma 3: There exists an integer δ, $0 \leq \delta \leq n-k$ for which an optimal schedule exist in which

  $$\{1, \ldots, k-1, k+1, \ldots, k+\delta\} \rightarrow k \text{ and } k \rightarrow \{k+\delta+1, \ldots, n\}.$$

  Proof on the board

Single machine models: Total Tardiness

DP-algorithm for $1||\sum T_j$

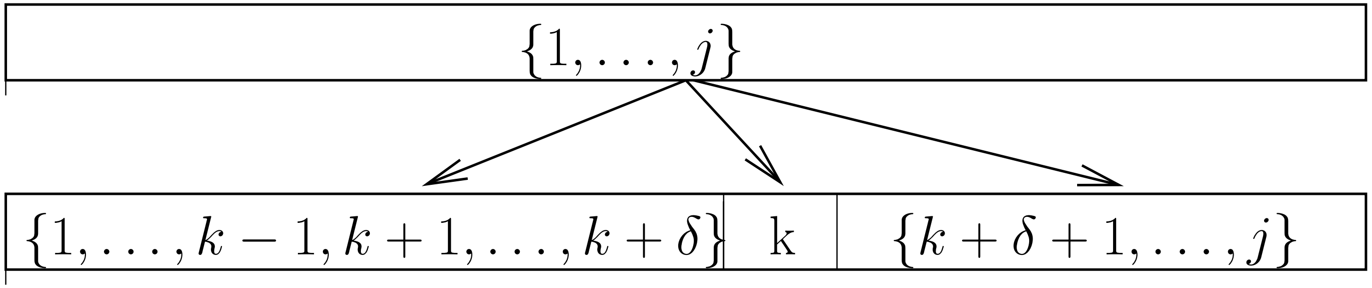
- Definition:
  - $F_j(t)$:= minimum criterion value for scheduling the first j jobs starting their processing at time t
- by Lemma 3 we get:
  there exists some $\delta \in \{1, \ldots, j\}$ such that $F_j(t)$ is achieved by scheduling

  1. first jobs $1, \ldots, k-1, k+1, \ldots, k+\delta$ in some order
  2. followed by job k starting at $t + \sum_{l=1}^{k+\delta} p_l - p_k$
  3. followed by jobs $k+\delta+1, \ldots, j$ in some order

  where $p_k = \max_{l=1}^{j} p_l$

Single machine models: Total Tardiness

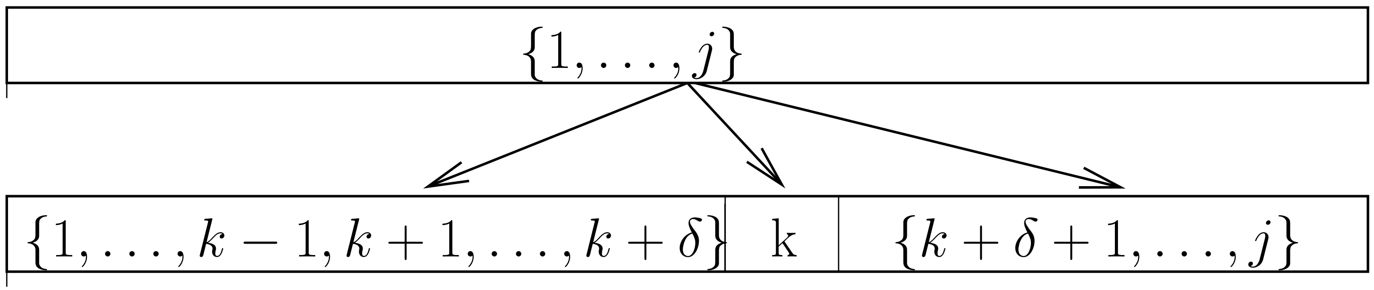
DP-algorithm for $1||\sum T_j$ (cont.)



- Definition:
  - $J(j,l,k) := \{i | i \in \{j, j+1, \ldots, l\}; p_i \leq p_k; i \neq k\}$

Single machine models: Total Tardiness

DP-algorithm for $1||\sum T_j$ (cont.)



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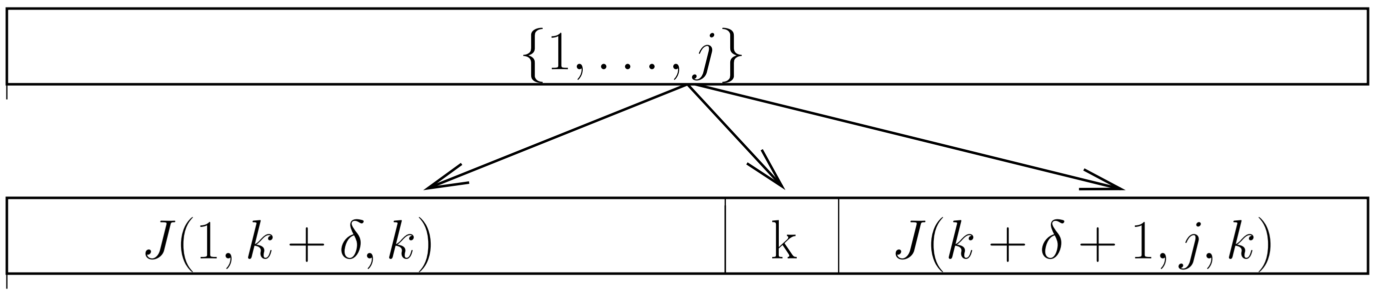
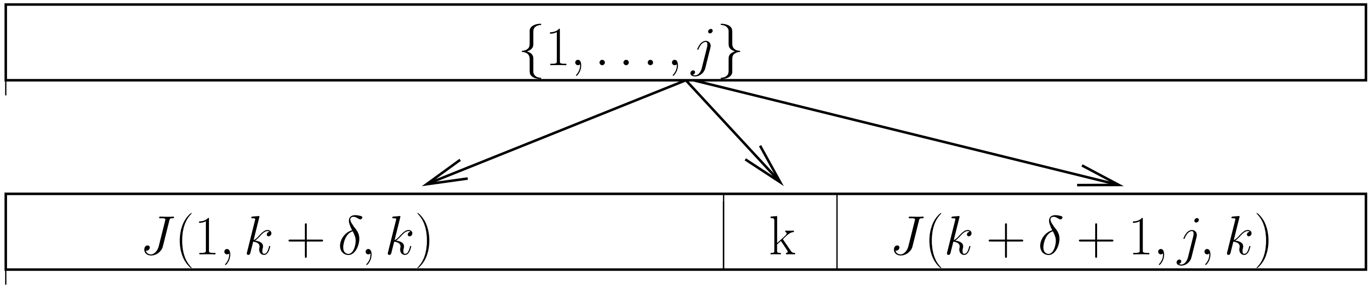


Single machine models: Total Tardiness

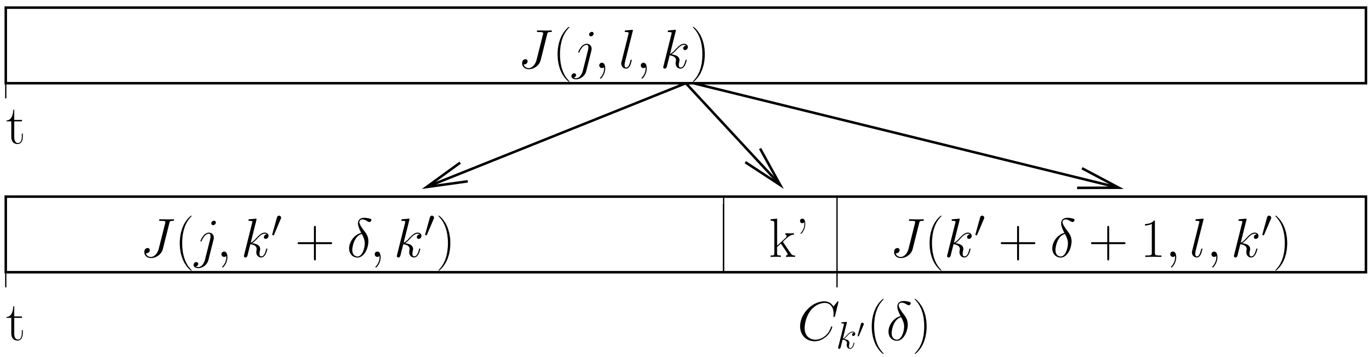
DP-algorithm for $1||\sum T_j$ (cont.)



- Definition:
  - $J(j,l,k) := \{i | i \in \{j, j+1, \ldots, l\}; p_i \leq p_k; i \neq k\}$
  - $V(J(j,l,k),t) :=$ minimum criterion value for scheduling the jobs from $J(j,l,k)$ starting their processing at time t

Single machine models: Total Tardiness

DP-algorithm for $1||\sum T_j$ (cont.)



- we get:

$$\begin{aligned} V(J(j,l,k),t) = \min_\delta \ \{ & V(J(j,k'+\delta,k'),t) \\ & + \max\{0, C_{k'}(\delta) - d_{k'}\} \\ & + V(J(k'+\delta+1,l,k'), C_{k'}(\delta)))\} \end{aligned}$$

where $p_{k'} = \max\{p'_j | j' \in J(j,l,k)\}$ and
$C_{k'}(\delta) = t + p_{k'} + \sum_{j' \in V(J(j,k'+\delta,k')} p_{j'}$

- $V(\emptyset, t) = 0$, $V(\{j\}, t) = \max\{0, t + p_j - d_j\}$

Single machine models: Total Tardiness

DP-algorithm for $1||\sum T_j$ (cont.)

- optimal value of $1||\sum T_j$ is given by $V(\{1,\ldots,n\},0)$
- complexity:
  - at most $O(n^3)$ subsets $J(j,l,k)$
  - at most $\sum p_j$ values for t
  - each recursion (evaluation $V(J(j,l,k),t)$) costs $O(n)$ (at most n values for δ)

total complexity: $O(n^4 \sum p_j)$ (pseudopolynomial)