Classification - Examples

- $1|r_j|C_{max}$
  - given: n jobs with processing times $p_1, \ldots, p_n$ and release dates $r_1, \ldots, r_n$
  - jobs have to be scheduled without preemption on one machine taking into account the earliest starting times of the jobs, such that the makespan is minimized
  - $n = 4$, $p = (2, 4, 2, 3)$, $r = (5, 4, 0, 3)$

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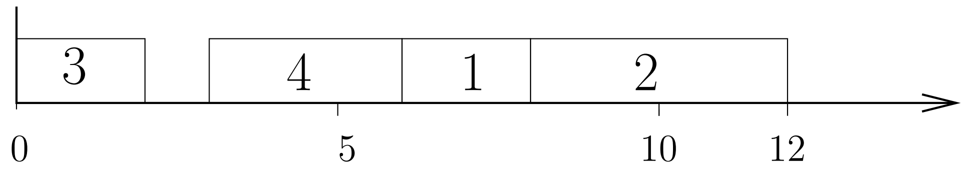

Feasible Schedule with $C_{max} = 12$ (schedule is optimal)

Classification - Examples

- $F2||\sum w_j T_j$
  - given n jobs with weights $w_1, \dots, w_n$ and due dates $d_1, \dots, d_n$
  - operations (i, j) with processing times $p_{ij},\ i = 1, 2;\ j = 1, \dots, n$
  - jobs have to be scheduled on two machines such that operation $(2, j)$ is schedules on machine 2 and does not start before operation $(1, j)$, which is scheduled on machine 1, is finished and the total weighted tardiness is minimized
  - $n = 3,\ p = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 4 & 1 \end{pmatrix},\ w = (3, 1, 5),\ d = (6, 8, 4)$

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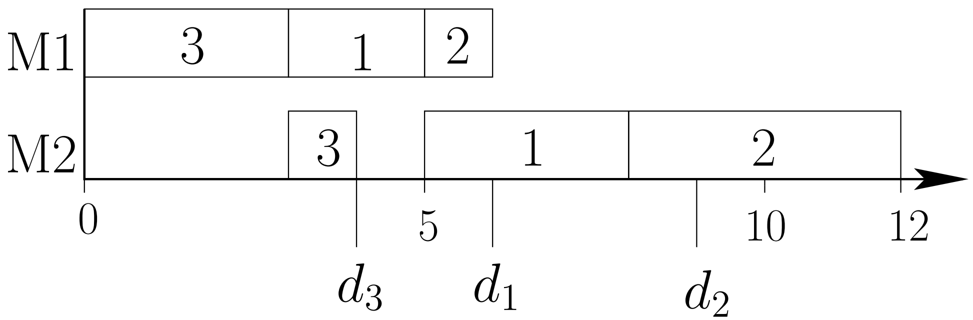


$$\sum w_jT_j = 3(8-6) + (12-9) \\ +5(4-4) = 9$$

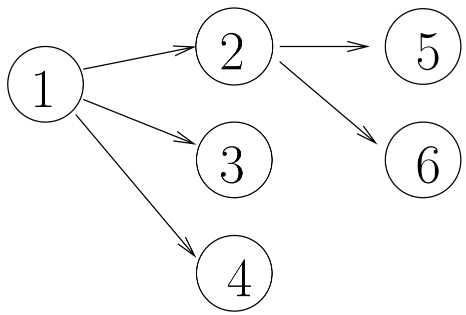
Classes of Schedules

- Nondelay Schedules:
  A feasible schedule is called a nondelay schedule if no machine is kept idle while a job/an operation is waiting for processing

  Example: $P3|prec|C_{max}$

  $n = 6$

  $p = (1, 1, 2, 2, 3, 3)$



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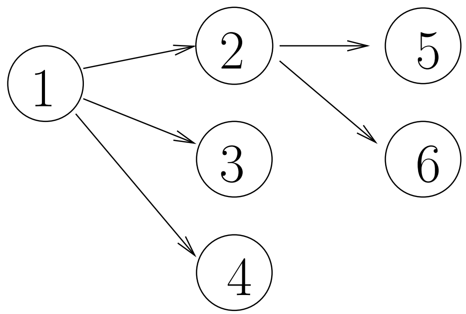
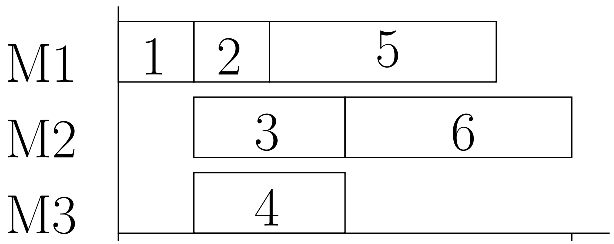
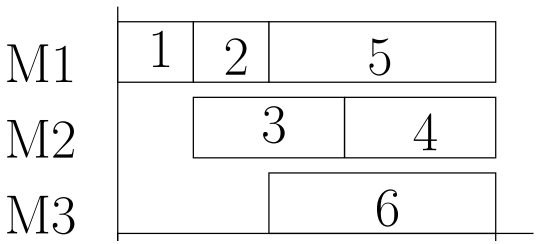


Best nondelay:



Optimal



Classes of Schedules

Remark: restricted to non preemptive schedules

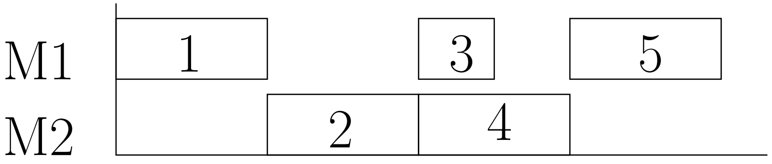
- Active Schedules:
  A feasible schedule is called active if it is not possible to construct another schedule by changing the order of processing on the machines and having at least one job/operation finishing earlier and no job/operation finishing later.
- Semi-Active Schedules:
  A feasible schedule is called semi-active if no job/operation can be finishing earlier without changing the order of processing on any one of the machines.

Classes of Schedules

Examples of (semi)-active schedules:

Prec: $1 \rightarrow 2$; $2 \rightarrow 3$

not semi-active



Classes of Schedules

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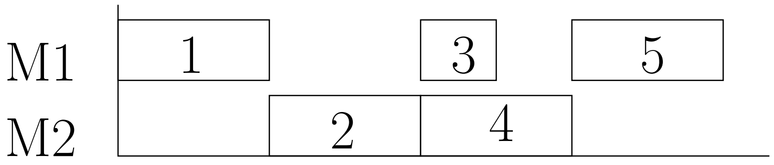
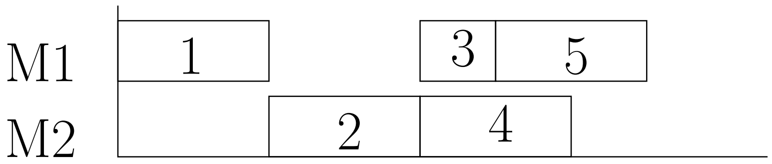


semi-active



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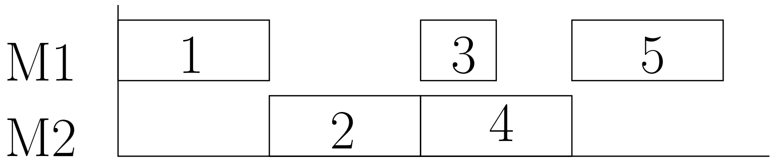
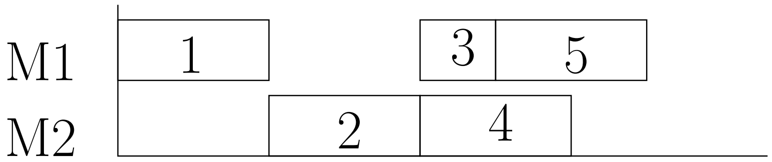
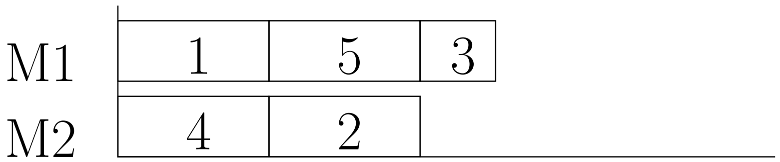


semi-active



active

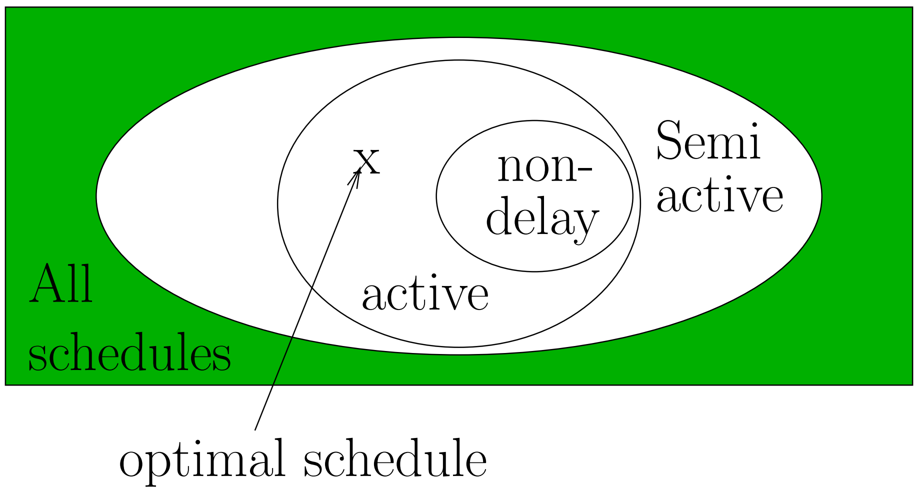


Classes of Schedules

Properties:

- every nonpreemptive nondelay schedule is active
- every active schedule is semiactive
- if the objective criterion is regular, the set of active schedules contains an optimal schedule (regular = non decreasing with respect to the completion times)

Summary:



Research topics for Scheduling

- determine boarder line between polynomially solvable and NP-hard models
- for polynomially solvable models
  - find the most efficient solution method (low complexity)
- for NP-hard models
  - develop enumerative methods (DP, branch and bound, branch and cut, ...)
  - develop heuristic approached (priority based, local search, ...)
  - consider approximation methods (with quality guarantee)

Intermezzo: Complexity Theory

- mathematical framework to study the difficulty of algorithmic problems

Notations/Definitions

- problem: generic description of a problem (e.g. $1||\sum C_j$)
- instance of a problem: given set of numerical data (e.g. $n, p_1, \ldots, p_n$)
- size of an instance I: length of the string necessary to specify the data (Notation: $|I|$)
  - binary encoding: $|I| = n + \log(p_1) + \ldots + \log(p_n)$
  - unary encoding: $|I| = n + p_1 + \ldots + p_n$

Intermezzo: Complexity Theory

Notations/Definitions

- efficiency of an algorithm: upper bound on number of steps depending on the size of the instance (worst case consideration)
- big O-notation: for an $O(f(n))$ algorithm a constant $c > 0$ and an integer n_0 exist, such that for an instance I with size $n = |I|$ and $n \geq n_0$ the number of steps is bounded by $cf(n)$
  Example: $7n^3 + 230n + 10\log(n)$ is $O(n^3)$
- (pseud)polynomial algorithm: $O(p(|I|))$ algorithm, where p is a polynomial and I is coded binary (unary)
  Example: an $O(n\log(\sum p_j))$ algorithm is a polynomial algorithm and an $O(n\sum p_j)$ algorithm is a pseudopolynomial algorithm

Intermezzo: Complexity Theory

Classes $\mathcal{P}$ and $\mathcal{NP}$

- a problem is (pseudo)polynomial solvable if a (pseudo)polynomial algorithm exists which solves the problem
- Class $\mathcal{P}$: contains all decision problems which are polynomial solvable
- Class $\mathcal{NP}$: contains all decision problems for which - given an 'yes' instance - the correct answer, given a proper clue, can be verified by a polynomial algorithm

Remark: each optimization problem has a corresponding decision problem by introducing a threshold for the objective value (does a schedule exist with objective smaller k?)

Intermezzo: Complexity Theory

Polynomial reduction

- a decision problem P polynomially reduces to a problem Q, if a polynomial function g exists that transforms instances of P to instances of Q such that I is a 'yes' instance of P if and only is $g(I)$ is a 'yes' instance of Q
  Notation: $P \propto Q$

NP-complete

- a decision problem $P \in \mathcal{NP}$ is called NP-complete if all problems from the class $\mathcal{NP}$ polynomially reduce to P
- an optimization problem is called NP-hard if the corresponding decision problem is NP-complete

Intermezzo: Complexity Theory

Examples of NP-complete problems:

- SATISFIABILITY: decision problem in Boolean logic, Cook in 1967 showed that all problems from $\mathcal{NP}$ polynomially reduce to it
- PARTITION:
  - given n positive integers $s_1, \ldots, s_n$ and $b = 1/2 \sum_{j=1}^{n} s_j$
  - does there exist a subset $J \subset I = \{1, \ldots, n\}$ such that

$$\sum_{j \in J} s_j = b = \sum_{j \in I \setminus J} s_j$$

Intermezzo: Complexity Theory

<u>Examples of NP-complete problems (cont.)</u>:

- 3-PARTITION:
  - given $3n$ positive integers $s_1, \ldots, s_{3n}$ and b with $b/4 < s_j < b/2,\ j = 1, \ldots, 3n$ and $b = 1/n \sum_{j=1}^{3n} s_j$
  - do there exist disjoint subsets $J_i \subset I = \{1, \ldots, 3n\}$ such that

$$\sum_{j \in J_i} s_j = b; \quad i = 1, \ldots, n$$

Intermezzo: Complexity Theory

Proofing NP-completeness

If an NP-complete problem P can be polynomially reduced to a problem $Q \in \mathcal{NP}$, than this proves that Q is NP-complete (transitivity of polynomial reductions)

Example: $PARTITION \propto P2||C_{max}$
Proof: on the board

Famous open problem: Is $\mathcal{P} = \mathcal{NP}$?

- solving one NP-complete problem polynomially, would imply $\mathcal{P} = \mathcal{NP}$

Single machine models

Observation:

- for non-preemptive problems and regular objectives, a sequence in which the jobs are processed is sufficient to describe a solution

Dispatching (priority) rules

- static rules - not time dependent
  e.g. shortest processing time first, earliest due date first
- dynamic rules - time dependent
  e.g. minimum slack first (slack= $d_j - p_j - t$; t current time)
- for some problems dispatching rules lead to optimal solutions

Single machine models: $1||\sum w_j C_j$

<u>Given</u>:

- n jobs with processing times $p_1, \ldots, p_n$ and weights $w_1, \ldots, w_n$

Consider case: $w_1 = \ldots = w_n (= 1)$:

Single machine models: $1||\sum w_jC_j$

Given:

- n jobs with processing times $p_1, \ldots, p_n$ and weights $w_1, \ldots, w_n$

Consider special case: $w_1 = \ldots = w_n (= 1)$:

- SPT-rule: shortest processing time first
- Theorem: SPT is optimal for $1||\sum C_j$
  Proof: by an exchange argument (on board)
- Complexity: $O(n \log(n))$

Single machine models: $1||\sum w_j C_j$

General case

- WSPT-rule: weighted shortest processing time first, i.e. sort jobs by increasing p_j/w_j-values
- Theorem: WSPT is optimal for $1||\sum w_j C_j$
  Proof: by an exchange argument (exercise)
- Complexity: $O(n \log(n))$

Further results:

- $1|tree|\sum w_j C_j$ can be solved by in polynomial time ($O(n \log(n))$)
  (see [Brucker 2004])
- $1|prec|\sum C_j$ is NP-hard in the strong sense
  (see [Brucker 2004])

Single machine models: $1|prec|f_{max}$

Given:

- n jobs with processing times $p_1, \ldots, p_n$
- regular functions $f_1, \ldots, f_n$
- objective criterion $f_{max} = \max\{f_1(C_1), \ldots, f_n(C_n)\}$

Observation:

- completion time of last job $= \sum p_j$

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Method

- plan backwards from $\sum p_j$ to 0
- from all available jobs (jobs from which all successors have already been scheduled), schedule the job which is 'cheapest' on that position

Single machine models: $1|prec|f_{max}$

S set of already scheduled jobs (initial: $S = \emptyset$)

J set of all jobs, which successors have been scheduled (initial: all jobs without successors)

t time where next job will be completed (initial: $t = \sum p_j$)

<u>Algorithm $1|prec|f_{max}$ (Lawler's Algorithm)</u>

REPEAT

select $j \in J$ such that $h_j(t) = \min_{k \in J} f_k(t)$;

schedule j such that it completes at t;

add j to S, delete j from J and update J;

$t := t - p_j$;

UNTIL $J = \emptyset$.

Single machine models: $1|prec|f_{max}$

- Theorem: Algorithm $1|prec|f_{max}$ is optimal for $1|prec|f_{max}$
  Proof: on the board
- Complexity: $O(n^2)$