Study of Decentralized Distribution Systems: Part II - Applications[1]

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Abstract

Anupindi, Bassok, and Zemel (1999) developed a general framework for the analysis of decentralized distribution systems. Several key ideas illustrated in that paper include a (i) "coopetitive" framework for the sequential decisions of inventory and allocation, (ii) the notion of *claims* that separate the ownership (with decision rights) and the location of inventories in the system, (iii) the existence of allocation mechanisms that achieve the first best solution, and (iv) meta-core as a criteria for the implementability of the solution concepts. The purpose of this paper is to explore, within the context of the general framework, the strategic role of some simple and intuitive allocation mechanisms in duopoly and oligopoly models. For the duopoly model, we develop sufficient conditions for the uniqueness of a pure strategy Nash Equilibrium and propose some simple allocation rules which exhibit unique Nash Equilibria. We then develop necessary and sufficient conditions under which a transfer pricing mechanism achieves first best. For a symmteric oligopoly model, we develop sufficient conditions for the existence of a unique Nash Equilibirum. Finally, for two specific oligopoly models (including the symmetric case) we show the existence of allocation policies that achieve the first best.

1 Introduction

A common problem in distribution systems that face uncertain demand is temporary imbalance of stocks. For example, one agent may be out-of-stock while another has excess stock. Clearly customers at the agent with a stock-out are unsatisfied. Exchange of stocks between these agents improves overall customer service. But who gets the benefits of this exchange? The agent who had excess stock or the agent who had excess demand? In general, how do we devise incentive mechanisms to facilitate this exchange in decentralized systems? How do these incentive mechanisms impact the strategic inventory deployment decision of the players involved? How does the performance of the decentralized system compare to a centrally coordinated system?

Anupindi, Bassok and Zemel (1999) (henceforth referred to as ABZ99) develop a general framework for the analysis of such systems. They demonstrate the existence of incentive mechanisms that give the centrally coordinated (first best) solution. Several key ideas illustrated in that paper include (i) a co-opetitive framework for the sequential decisions of inventory and allocation, (ii) the notion of *claims* that separate the ownership (with decision rights) and the location of inventories in the system, (iii) the existence of allocation mechanisms that achieve the first best solution, and (iv) meta-core as a criteria for the implementability of the solution concepts.

The purpose of this paper is to illustrate the framework developed in ABZ99 for a general duopoly system and two oligopoly models. In doing so, we are able to sharpen several results. For example, for a duopoly model, we develop sufficient conditions for the uniqueness of a pure strategy Nash Equilibrium and propose some simple allocation rules which exhibit unique Nash Equilibria. We then present simple allocation rules under which the Nash Equilibrium is unique. We also explore whether some of these simpler allocation rules (e.g., transfer pricing rules) achieve the first best solution. Specifically, for a duopoly system, we develop necessary and sufficient conditions under which a transfer pricing rule gives the first best. For a symmetric oligopoly model, we develop sufficient conditions for the existence of a unique Nash Equilibirum. Subsequently, for two oligopoly models (including the symmetric case), we explore the performance of some intuitive allocation mechanisms and show that a transfer pricing rule does not give the first best solution even under simplified assumptions.

According to ABZ99, work on decentralized distribution systems can be classified along three attributes, namely *search* (customer or retailer driven), *system* (duopoly or oligopoly), and *interaction* (horizontal and / or vertical). A detailed literature review based on this criteria can be found in ABZ99. For completeness, here we briefly discuss the essentials.

Several papers have analyzed a duopoly system. Parlar (1988) and Anupindi and Bassok (1999) analyze duopoly models that consider customer-driven search. Customer-driven search implies a specific allocation rule for surplus generated from satisfying excess demand at one retailer using excess stock at the other. As we will show, this allocation policy does not give the first best solution. Two papers – van Mieghem (1998) and Rudi, Kapur, and Pyke (1998) – consider retailer-driven search in duopoly systems. Given the context of a manufacturer and a sub-contractor, the search in van Mieghem (1998) can be considered uni-directional. He evaluates mechanisms that achieve the first best solution and demonstrates that a simple transfer pricing mechanism does not achieve the first best. Lippman and McCardle (1997) analyze a competitive newsboy (oligopoly) model. The allocation rule for excess stock at one location to satisfy excess demands at another is very similar to the one considered by Parlar and hence can be considered customer-driven. They then examine the relationship between equilibrium inventory levels and the allocation rules and provide conditions under which there is a unique equilibrium. All of the above papers, except Anupindi and Bassok (1999), analyze only the horizontal interaction between the retailers.

The rest of the paper is organized as follows. In Section 2 we briefly summarize the key concepts of the general framework developed in ABZ99. In Section 3 we analyze the general duopoly model. In section 4 we analyze two oligopoly models. Finally, we conclude in section 5.

2 Key Concepts of the General Framework

In this section, we summarize the key concepts from the general framework developed in ABZ99.

Consider a set $\mathcal{N} = \{1, \cdots, N\}$ of retailers of a common product, each of which faces his own demand and manages his own inventory and sales. We use the well known framework of the single period, single item *newsvendor problem*. Specifically, for each $n \in \mathcal{N}$, let c_n, r_n, and v_n be the unit cost, unit revenue, and unit salvage value, respectively. In addition, each retailer $n \in \mathcal{N}$ is faced with its own demand distribution, D_n , with $\vec{D} = (D_1, \cdots, D_N)$. We assume that the demand profile $\vec{D}$ is distributed according to a continuous joint cumulative density function (CDF) $F(\vec{D})$, with a corresponding set of marginal (not necessarily independent) CDF's denoted $F_n(D_n)$. Before the actual demand D_n is realized each retailer $n \in \mathcal{N}$ orders (and pays for) a certain quantity of stock X_n. We denote the profile of stock ordered by $\vec{X} = \{X_1, \cdots, X_N\}$. The actual sales by retailer n is denoted $S_n = \min\{X_n, D_n\}$, leftover inventories for salvage $H_n = \max\{X_n - D_n, 0\}$, and the excess demand is $E_n = \max\{D_n - X_n, 0\}$.

In addition to local stocks, the distribution system consists of a set $\mathcal{W} = \{1, \cdots, W\}$ of *centralized warehouses*. For each warehouse $w \in \mathcal{W}$, we let c_w be the purchasing cost at warehouse w, v_w the salvage value of excess stocks at warehouse w, and Y_w stock position at warehouse w with $\vec{Y} = (Y_1, \cdots, Y_W)$. Finally, we also define $t_{i,n}$ as the per unit additional shipping, handling or discounting costs associated with meeting demand at retailer n from location (retailer or warehouse) i for each $i \in \mathcal{W} \cup \mathcal{N}$, $n \in \mathcal{N}$. Finally, $\beta_{i,n}$ represents the fraction of customers at retailer n who will accept service out of location i.

Finally, we assume a system with *pooling of residuals*; that is, that each agent covers his own demand first and only then the residual inventory is shipped to satisfy any unmet demand at other locations.

2.1 First Best Solution

The first best solution is obtained by solving the inventory and shipping decisions for the distribution system assuming a single decision maker. The model can be formulated as a two-stage stochastic program; see ABZ99 for details. Here we just write $J^c_{\mathcal{N}}(\vec{X}, \vec{Y})$ as the expected value of the profits for a given inventory profile $(\vec{X}, \vec{Y})$ with $\Pi^c_{\mathcal{N}} = \max_{\vec{X},\vec{Y}} J^c_{\mathcal{N}}(\vec{X}, \vec{Y})$ representing the *first best profits* and $(\vec{X}^{c*}, \vec{Y}^{c*}) = \text{argmax}_{\vec{X},\vec{Y}} J^c_{\mathcal{N}}(\vec{X}, \vec{Y})$ as the *first best solution*.

2.2 Claims

The common inventory $\vec{Y}$ is **claimed** *if each unit of inventory that comprises $\vec{Y}$ is separately owned (regardless of its location) by some retailer.* Let $Y_{w,n}$ denote the inventory at centralized location $w \in \mathcal{W}$ claimed by retailer n. We require that $\sum_{n=1}^{N} Y_{w,n} = Y_w$, for all $w \in \mathcal{W}$. Claims establishes ownership of each unit of inventory in the system. They are established by purchasing inventory before demand is realized and entitle the owner, in turn, to determine its use ex post.

2.3 Co-opetitive Framework

Assuming that all common inventory is claimed, there are two classes of decisions to be made in sequence, namely inventory and shipping/allocation as follows:

1. *Inventory Decisions*: Before the demand profile $\vec{D}$ is realized:

   (a) Each agent, $n \in \mathcal{N}$ must determine his inventory position, $(X_n, Y_{1,n}, \ldots, Y_{W,n}) \equiv \vec{Z}_n$.

2. *Shipping Decisions*: In addition, the following decisions need to be made after the demand profile is realized:

   (a) the deployment of the residual and common available inventory stocks towards the various residual demands at the respective locations after demand profile is realized, and

   (b) the allocation of the excess profits generated by sales at the various retailers due to the shipping decisions for a given realized demand profile.

While the allocation of profits are done after the demand profile is realized, the players need to reach an agreement regarding the allocation mechanism before the demand profile is realized. Finally, observe that the shipping decision and the resulting allocation of profits is cooperative whereas the inventory decision is non-cooperative. As solution concepts, for the former we use the notion of core and for the latter the Nash Equilibrium. We briefly discuss these next.

2.3.1 Cooperative Shipping / Allocation Decision

Let $[\mathbf{Z}] = (\vec{Z}_1, \ldots, \vec{Z}_n)$ denote the inventory position of all players. For each coalition $\mathcal{S} \subseteq \mathcal{N}$, let $W^*_{\mathcal{S}}([\mathbf{Z}], \vec{D})$ be the maximal excess profit available to members of $\mathcal{S}$ (in addition to what could be achieved without pooling) for a specific demand realization. $W^*_{\mathcal{S}}([\mathbf{Z}], \vec{D})$ is the optimal value of a linear program which decides the optimal shipping pattern; see ABZ99 for details. There are two types of allocation games to consider.

• **Snapshot Allocation Games** ($\mathbf{SAG}(\vec{D})$): Let $\vec{\alpha}([\mathbf{Z}], \vec{D})$ be the allocation of profits to each player for a given $[\mathbf{Z}]$ and $\vec{D}$. In $\mathbf{SAG}(\vec{D})$, we seek allocation mechanisms in a core where the value of the game is derived by optimizing the shipping decisions for a given realization of the random demands. Thus, $\vec{\alpha}([\mathbf{Z}], \vec{D})$ should satisfy:

$$\sum_{j \in \mathcal{S}} \alpha_j([\mathbf{Z}], \vec{D}) \geq W^*_{\mathcal{S}}([\mathbf{Z}], \vec{D}) \qquad \forall \mathcal{S} \subseteq \mathcal{N} \tag{1a}$$

$$\sum_{j \in N} \alpha_j([\mathbf{Z}], \vec{D}) = W^*_{\mathcal{N}}([\mathbf{Z}], \vec{D}) \tag{1b}$$

ABZ99 show that the core of $\mathbf{SAG}(\vec{D})$ is non-empty. Specifically allocations based on the dual price vector of the shipping decision are in the core. Clearly, an allocation rule with allocations in $\mathbf{SAG}(\vec{D})$ is renegotiation proof (Fudenberg & Tirole, 1991).

• **Allocation Games in Expectation** (**AGE**): In **AGE** we seek allocation mechanisms in a core where the value of the game is derived by taking expectation (over all possible realizations of the random demands) of the profits generated from the optimal shipping decisions for each set of realizations. That is, if $W^*_{\mathcal{S}}([\mathbf{Z}], \vec{D})$ represents the value of a $\mathbf{SAG}(\vec{D})$ for a specific demand vector $\vec{D}$ and for coalition $\mathcal{S}$, then the value of the allocation game in expectation (**AGE**) is given by

$$W^*_{\mathcal{S}}([\mathbf{Z}]) = \mathbf{E}_{\vec{D}} W^*_{\mathcal{S}}([\mathbf{Z}], \vec{D}).$$

In **AGE** we seek allocations $\alpha_j([\mathbf{Z}])$ such that

$$\sum_{j \in \mathcal{S}} \alpha_j([\mathbf{Z}]) \geq W^*_{\mathcal{S}}([\mathbf{Z}]) \qquad \forall \mathcal{S} \subseteq \mathcal{N} \tag{2a}$$

$$\sum_{j \in N} \alpha_j([\mathbf{Z}]) = W^*_{\mathcal{N}}([\mathbf{Z}]). \tag{2b}$$

Observe that an allocation rule with allocations in **AGE** is not renegotiation proof. Therefore, to implement such an allocation rule, we assume that the players agree to a "long-term" contract on the allocations and with no opportunity to renege regardless of the demand realization.

2.3.2 Non-Cooperative Inventory Decision

Given an agreed upon Allocation Rule m (denoted by AR-m), that belongs to the core of either $\mathbf{SAG}(\vec{D})$ or **AGE**, we denote the total profits of retailer n as

$$P^m_n([\mathbf{Z}], \vec{D}) = r_n S_n + v_n H_n - c_n X_n - \sum_{w=1}^{W} c_w Y_{w,n} + \alpha_n([\mathbf{Z}], \vec{D}).$$

Let $J_n^m([\mathbf{Z}]) = \mathbf{E}_{\vec{D}} P_n^m([\mathbf{Z}], \vec{D})$. Then the Nash Equilibrium (NE) of the inventory problem for a given allocation rule AR-m is given by

$$J_n^m([\mathbf{Z^{m*}}]) \geq J_n^m([\mathbf{Z^{m*}}]_{-n} \cup \vec{Z}_n) \qquad \forall \vec{Z}_n, \forall n \in \mathcal{N}. \tag{3}$$

That is, $\vec{Z}_n^{m*}$ is the optimal policy for retailer n for a given AR-m and actions of all other players.

2.4 Constructing Allocation Mechanisms

We summarize one important result developed in ABZ99 that allows us to construct new allocation mechanisms that are in the core ($\mathbf{SAG}(\vec{D})$ or **AGE**) and implement a desired Nash Equilibrium solution.

Theorem 2.1 *Consider an allocation policy AR-m with allocations $\alpha_n^m([\mathbf{Z}], \vec{D})$ to player $n \in \mathcal{N}$. Let $J_n^m(\cdot)$ be the profit function of player n under AR-m. We assume that $J_n^m(\cdot)$ are continuous and unimodular so that the existence of a Nash equilibrium in pure strategies is guaranteed. Let $[\mathbf{Z}]^{m*}$ be a Nash equilibrium in pure strategies. Then there exist constants $w_n([\mathbf{Z}]^{m*}, \vec{D})$ using which we can construct a new allocation policy, denoted by AR-$\tilde{m}$, with allocations*

$$\alpha_n^{\tilde{m}}([\mathbf{Z}], \vec{D}) = \alpha_n^m([\mathbf{Z}], \vec{D}) + w_n([\mathbf{Z}]^{m*}, \vec{D})$$

such that (i) $[\mathbf{Z}]^{\widetilde{m}*} = [\mathbf{Z}]^{m*}$ and (ii) AR-$\widetilde{m}$ is in core of $\mathbf{SAG}(\vec{D})$ or $\mathbf{AGE}$ whenever $[\mathbf{Z}] = [\mathbf{Z}]^{m*}$.

2.5 Criteria for Implementability of a Solution: Meta-Core

If the agents negotiate over the allocation policies, then we need to provide criteria for the choice of an allocation policy. We refer to this as the **meta-core**.

Consider a sub-coalition $\mathcal{S} \subset \mathcal{N}$. Suppose an allocation policy AR-m̂ is used by the members of this sub-coalition. Let $J_n^{\hat{m},\mathcal{S}}([\mathbf{Z}^{\hat{\mathbf{m}}*}])$ denote the payoff function of the Nash equilibrium game for this sub-coalition; we explicitly show the dependence the payoff function on the coalition $\mathcal{S}$. For an allocation policy AR-m̂, define the value of this coalition as follows:

$$U_{\mathcal{S}}(\hat{m}) = \sum_{k \in \mathcal{S}} J_k^{\hat{m},\mathcal{S}}([\mathbf{Z}^{\hat{\mathbf{m}}*}]).$$

Clearly the value of this sub-coalition will be a function of the allocation policy AR-m. Let allocation policy AR-$m_{\mathcal{S}}$ maximize the value of this sub-coalition. That is,

$$U_{\mathcal{S}}^*(m_{\mathcal{S}}) = \max_{\hat{m}} U_{\mathcal{S}}(\hat{m}).$$

Now consider all allocation policies AR-m that are in the core for the grand coalition $\mathcal{N}$ such that the payoffs of the individual players are *Pareto optimal*. Then the set of allocation policies that are candidates for implementation should satisfy the following conditions:

$$\sum_{n \in \mathcal{S}} J_n^{m,\mathcal{N}}([\mathbf{Z^m}]) \geq U_{\mathcal{S}}^*(m_{\mathcal{S}}) \qquad \forall \mathcal{S} \subset \mathcal{N}. \tag{4}$$

If an allocation policy AR-m satisfies (4), then we say that it belongs to the meta-core.

3 A Duopoly Model with Claims

We now illustrate the general framework using a duopoly model with claims. In doing so, we first develop sufficient conditions for the uniqueness of a NE. We then suggest several intuitive allocation policies and illustrate that the NE is unique for these policies. We also develop necessary and sufficient conditions that ensure that a transfer pricing mechanism will give the first best solution.

Let S_n, H_n, and E_n represent respectively the sales, excess leftover inventory, and excess demand at retailer n before any transhipments. That is,

$$S_n = \min\{X_n, D_n\}$$
$$H_n = \max\{X_n - D_n, 0\}$$
$$E_n = \max\{D_n - X_n, 0\}.$$

Observe that $X_n = S_n + H_n$ and $D_n = S_n + E_n$. We also define

$S = S_1 + S_2$, $H = H_1 + H_2$, $E = E_1 + E_2$, $D = D_1 + D_2$, and $X = X_1 + X_2$.

The total sales resulting from the transhipments is given by $\min(Y, H, E)$. To allocate the profits that result from the transhipments, one may define various allocation rules (AR) each of which result in a different allocation of residual profits for retailer n. An AR could be *a posteriori* (depend on variables, e.g., sales, excess stock, etc., that are function of the realization of demands) or *a priori* (based on some prespecified ratios). We now list some possible allocation rules that define the fraction of total surplus, Θ_n^m, allocated to player $n = 1, 2$.

AR-γ Allocate in proportion to some predetermined fraction, γ_n: $\Theta_n^\gamma = \gamma_n$, $\gamma_n \in [0, 1]$, $\gamma_1 + \gamma_2 = 1$.

AR-e Allocate in proportion to excess demand, E_n: $\Theta_n^e = \frac{E_n}{E}$;

AR-h Allocate in proportion to excess stock, H_n: $\Theta_n^h = \frac{H_n}{H}$;

(The letter m in AR-m signifies the allocation variable.) AR-γ is a simple rule. AR-h and AR-e are intuitive since the former gives all credit to the player who satisfies demand whereas the latter gives all the credit to the player who generates the demand. Several other allocation mechanisms can be outlined; e.g., allocations in the ratio of realized sales, realized demands, etc. Alternately, simple transfer pricing rules can be devised. We will show later that a transfer pricing scheme can be devised as a convex combination of the allocation rules AR-h and AR-e. However, do the retailers have an incentive to participate in this exchange?

Proposition 3.1 *In a duopoly system, any allocation mechanism that gives non-negative residual profits to each player is in the core of* $\mathbf{SAG}(\vec{D})$.

This is straightforward to see since each sub-coalition is a singleton and constraints (1) are easily satisfied. Hence, in a duopoly, all the allocation mechanisms listed earlier are in the core of $\mathbf{SAG}(\vec{D})$ and **AGE**. While AR-γ is an *a priori* allocation rules, AR-e and AR-h are *a posteriori* allocation rules. The latter, however, can be converted to *a priori* rules by replacing the corresponding random variables (E_n, H_n) by their expectations. In the rest of this section, we will focus our attention on allocation rules AR-h and AR-e. Allocation rules AR-h and AR-e are attractive since they are intuitive.

3.1 Uniqueness of Nash Equilibrium

We begin by presenting a general result for the uniqueness of NE in a duopoly where the strategy space of a player is one-dimensional.

Theorem 3.1 *Suppose that the payoff function of player* i, $J_i(\vec{X})$ *is concave in* X_i *and twice differentiable. Define the following function:*

$$R_i(\vec{X}) = -\frac{\frac{\partial^2 J_i(\vec{X})}{\partial X_i \partial X_j}}{\frac{\partial^2 J_i(\vec{X})}{\partial {X_i}^2}}.$$

Then there exists a unique pure strategy NE if either condition D1 or D2 holds:

D1: $|R_1(\vec{X}) \cdot R_2(\vec{X})| < 1$

D2: $|R_1(\vec{X})| > 1$ *and* $|R_2(\vec{X})| > 1$.

Proof: Notice that $R_i(\cdot)$ is the slope of the reaction function of player i. To ensure uniqueness, we need to ensure that the respective reaction functions intersect only once. Uniqueness of the NE under condition D1 follows from Proposition 4.1 of Başar and Jan Olsder (1995). Thus Theorem 3.1 extends the result of Başar and Jan Olsder (1995) when the strategy space each player is one-dimensional. To see the uniqueness under condition D2, draw both the reaction functions with X_1 along the x-axis and X_2 along the y-axis; that is, draw $R_2(X_1)$ and $R_1^{-1}(X_2)$. These functions will intersect only once if slope of one dominates the other. Condition D2 ensures this. ∎

For the rest of this section, we make the following assumptions:

A1: The unit revenue at location i is larger than the unit revenue at location j less the transhipment cost from i to j; that is $r_i > r_j - t_{i,j}$.

A2: The unit cost of transhipment from location i to j exceeds the corresponding differential in salvage values; that is $t_{j,i} > v_i - v_j$.

A3: *All* customers at retailer n will accept service out of location i if need be; that is, $\beta_{i,n} = 1$ for all $i, n \in \{1, 2\}$.

A4: There is no pooled inventory; that is $\vec{Y} = 0$.

Assumption A1 ensures it is profitable to satisfy local demand first. Assumption A2 ensures that it is not profitable to salvage at an alternate location. Assumptions A3 is made for convenience of analysis. Assumption A4 ensures that the strategy space of a player is one-dimensional. We could relax Assumption A4 if we assume that there is no local inventory.

Recall that $J_n^m(\vec{X})$ represents the expected profits of player n when allocation rule AR-m is used. It is useful to write the payoff function of player n as consisting of two parts: a direct profit (DP_n) earned by satisfying local demand as much as possible and an indirect profit (IP_n^m) which represents the allocation of excess profits to player n under allocation rule m generated from the exchange of goods between the two players. Thus,

$$J_n^m(\vec{X}) = \mathbf{E}[DP_n(X_n, D_n)] + \mathbf{E}[IP_n^m(\vec{X}, \vec{D})], \tag{5}$$

where

$$DP_n(X_n, D_n) = \mathbf{E}[r_n \min(X_n, D_n) + v \max(X_n - D_n, 0)] - c_n X_n$$

and $IP_n^m(\vec{X},\vec{D}) = \Theta_n^m W_{\mathcal{N}}^*(\vec{X},\vec{D})$. $W_{\mathcal{N}}^*(\vec{X},\vec{D})$ is the total surplus generated from the exchange of goods between the two players. If $q_{i,j}$ represents the quantity shipped from player i to j then

$$W_{\mathcal{N}}^*(\vec{X},\vec{D}) = \max_{q_{1,2},q_{2,1}} (r_2 - v_1 - t_{1,2})q_{1,2} + (r_1 - v_2 - t_{2,1})q_{2,1} \tag{6a}$$

subject to

$$q_{1,2} \leq H_1 \tag{6b}$$

$$q_{2,1} \leq H_2 \tag{6c}$$

$$q_{1,2} \leq E_2 \tag{6d}$$

$$q_{2,1} \leq E_1 \tag{6e}$$

$$q_{1,2}, q_{2,1} \geq 0. \tag{6f}$$

Let $\vec{X}^{m*}$ be the NE for AR-m satisfying (3) with $\vec{Z}_n = X_n$.

In the remaining part of this section, we will assume specific allocation policies. To this extent, we need to write the payoff functions explicitly. We make the following additional assumption for convenience:

A5: Demands across the retailers are independent.

Applying this result to duopoly with AR-h and AR-e, we get the following result.

Proposition 3.2 *Given Assumptions A1-A5, there exists a unique pure strategy NE for the duopoly model with claims under both AR-h and AR-e.*

Assumption A1 ensures that the game of *pooling with residuals* is indeed optimal. It also ensures that the payoff function of player i is concave in X_i. In addition, Assumption A2 ensures that the payoff function under AR-e is concave. It is then straightforward to show that Condition D1 of Theorem 3.1 is satisfied under each of the allocation rules.

We now focus our attention to study the effect of allocation mechanisms on inventory decisions.

3.2 Effect of Allocation Mechanism on Inventory Decisions

We will now discuss the effect of allocation mechanisms AR-h and AR-e on the equilibrium inventory decisions of the two players. In addition, we examine whether the first best solution can be achieved. Recall that $\vec{X}^{c*}$ represents the first best inventory decision. This is obtained by solving the general duopoly model assuming a single decision maker controls the system. Yet another benchmark is the unilateral inventory of each player should they decide not to share any stocks after demand realizations; in this case there is no interaction between the two players. Subsequently, each player behaves like an independent newsvendor and decides on stock level X_i^{nb} given by:

$$F_i(X_i^{nb}) = \frac{r_i - c_i}{r_i - v_i}.$$

Proposition 3.3 *The equilibrium inventory in the decentralized system when profits are allocated in proportion to excess demand (AR-e) and the equilibrium inventory when the profits are allocated in proportion to excess inventory (AR-h) are respectively below and above the optimal inventory when the players behave as independent newsvendors. That is,* $\vec{X}^{e*} \leq \vec{X}^{nb} \leq \vec{X}^{h*}$.

How do these inventory levels compare with the first best solution? It is straightforward to show that the reaction functions of AR-e (respectively, AR-h) is dominated by (respectively, dominates) the corresponding first partial of the profit function in the centralized system (which gives the first best solution). This, however, does not imply any dominance result for the corresponding equilibria. We can make more precise statements if we assume that the players are symmetric in terms of their cost and revenue parameters.

Proposition 3.4 *Assume that the players face identical procurement costs, sell at identical prices, and get identical salvage value for excess goods. We then have the following two cases:*

1. *Suppose, in addition, that transshipment costs are zero but demand densities are non-identical. Then the total inventory in the centralized system (X^{c*}) is above (respectively, below) the total inventory under AR-e (respectively, AR-h); that is,*

$$X_1^{e*} + X_2^{e*} \leq X^{c*} \leq X_1^{h*} + X_2^{h*}.$$

2. *Suppose, in addition, that transhipment costs are identical but strictly positive and demands identically distributed, then the inventory vector in the centralized system is above (respectively, below) the equilibrium inventory vector under AR-e (respectively, AR-h); that is,*

$$\vec{X}^{e*} \leq \vec{X}^{c*} \leq \vec{X}^{h*}.$$

That is, with AR-e (respectively, AR-h) the agents in a decentralized system under– (respectively, over–) invest compared to a centralized system. This is to be expected. Observe that under AR-h since all surplus profits goes to the player who satisfies demand, there is incentive to have excess stock and hence the higher inventory level. Conversely, under AR-e all surplus is given to the player with excess demand which creates an incentive to be the initiator of demand by incurring shortages. This prompts lower inventory investment.

Are there allocation mechanisms that induce cooperative behavior (i.e., achieve the first best solution)? Fortunately, the answer is yes. First define the following functions:

$$A_1(X_1, X_2) \equiv (r_1 - v_2 - t_{21}) \int_0^{X_2} [F_1(X_1 + X_2 - d_2) - F_1(X_1)] dF_2(d_2)$$
$$B_1(X_1, X_2) \equiv (r_2 - v_1 - t_{12}) \int_0^{X_1} \bar{F}_2(X_1 + X_2 - d_1) dF_1(d_1)$$
$$A_2(X_1, X_2) \equiv (r_1 - v_2 - t_{21}) \int_0^{X_2} \bar{F}_1(X_1 + X_2 - d_2) dF_2(d_2)$$
$$B_2(X_1, X_2) \equiv (r_2 - v_2 - t_{12}) \int_0^{X_1} [F_2(X_1 + X_2 - d_1) - F_2(X_2)] dF_1(d_1).$$

Let $A_i^c = A_i(X_1^{c*}, X_2^{c*})$ and $B_i^c = B_i(X_1^{c*}, X_2^{c*})$. Now,

Theorem 3.2 *Consider the following allocation rule, denoted by AR-(eh), using which*

- *allocate to player 2 a fraction $\gamma_1 \in [0,1]$ of the total surplus generated by transhipments from player 2 to player 1 and the remaining $(1-\gamma_1)$ to player 1.*
- *allocate to player 1 a fraction $\gamma_2 \in [0,1]$ of the total surplus generated by transhipments from player 1 to player 2 and the remaining $(1-\gamma_2)$ to player 2.*

Then, there exists $\gamma_1 \in [0,1]$ and $\gamma_2 \in [0,1]$ with

$$\gamma_1 = \frac{(B_2^c - A_2^c)B_1^c}{B_2^c A_1^c - B_1^c A_2^c} \tag{7}$$
$$\gamma_2 = \frac{(A_1^c - B_1^c)A_2^c}{B_2^c A_1^c - B_1^c A_2^c} \tag{8}$$

such that the equilibrium inventory decision in the decentralized system is first best (that is, $\vec{X}^{(eh)*} = \vec{X}^{c*}$) if and only if $\vec{X}^{c*} \le \vec{X}^{nb}$ or $\vec{X}^{c*} \ge \vec{X}^{nb}$.

The allocation rule AR-(eh) can also be interpreted as a *transfer pricing* mechanism. Suppose the transfer price for exchange of goods between player i and j is s_{ij}. That is, if j obtains goods from i, it pays s_{ij} per unit to i. Specifically, if $E_1 > 0$ and $H_2 > 0$, then player 1 receives goods from player 2. The marginal surplus generated is $r_1 - v_2 - t_{21}$. Then, player 1 pays $s_{21} = \gamma_1(r_1 - v_2 - t_{21})$ to player 2 for each unit of good. Similarly, if $E_2 > 0$ and $H_1 > 0$, the player 2 pays $s_{12} = \gamma_2(r_2 - v_1 - t_{12})$ for each unit of good it obtains from player 1.

Observe that the necessary and sufficient conditions in Theorem 3.2 for first best involve comparison of inventory decisions between a centralized system ($\vec{X}^{c*}$) and the independent newsvendor problem ($\vec{X}^{nb}$). In general, it is unclear if $\vec{X}^{c*}$ is uniformly above or below $\vec{X}^{nb}$. It is possible that $X_i^{c*} > X_i^{nb}$ but $X_j^{c*} < X_j^{nb}$, for $j \neq i$.[2] From Theorem 3.2 when this happens the first best is not achievable with non-negative transfer of surplus. If we allow for negative transfers, then the first best solution is achievable using the AR-(eh) policy. Negative transfers (i.e., either γ_1 or γ_2 is negative) imply that one of players subsidizes the other. Suppose $\gamma_1 < 0$. That is, not only does player 1 get all the surplus from any transhipment from player 2 to 1 but player 2 pays additional money to player 1. But this violates Proposition 3.1 and player 2 will not participate in the exchange of stocks. The situation, however, can be fixed by suitably modifying the allocation policy AR-(eh) using Theorem 2.1 as follows.

Observe that the allocation to each player under AR-(eh) is given by:

$$\alpha_1^{(eh)}(\vec{X}, \vec{D}) = (1 - \gamma_1)(r_1 - v_2 - t_{2,1}) \min\{H_2, E_1\} + (\gamma_2)(r_2 - v_1 - t_{1,2}) \min\{H_1, E_2\} \tag{9}$$

$$\alpha_2^{(eh)}(\vec{X}, \vec{D}) = (\gamma_1)(r_1 - v_2 - t_{2,1}) \min\{H_2, E_1\} + (1 - \gamma_2)(r_2 - v_1 - t_{1,2}) \min\{H_1, E_2\} \tag{10}$$

Assume γ_n is not constrained to be within $[0, 1]$. Then AR-(eh) gives the first best solution but some of the γ_n may be negative. To use Theorem 2.1 we need to define allocations based on the dual prices for the first best inventory decision ($\vec{X}^{c*}$). Let ν_1, ν_2, δ_1 and δ_2 be the dual prices corresponding to the constraints (6b)– (6e) for the shipping decision with inventory vector as $\vec{X}^{c*}$. Let $\alpha_n^d(\vec{X}^{c*}, \vec{D})$ be the allocation of surplus to player n when the dual allocation policy is used. Specifically,

$$\alpha_1^d(\vec{X}^{c*}, \vec{D}) = \nu_1 H_1^c + \delta_1 E_1^c \tag{11}$$

$$\alpha_2^d(\vec{X}^{c*}, \vec{D}) = \nu_2 H_2^c + \delta_2 E_2^c \tag{12}$$

where H_n^c and E_n^c represent the excess stock and excess demand at player n for first best inventories ($\vec{X}^{c*}$).

[2]This happens for example when only one–way transshipment is possible; that is, player 1 (or 2) satisfies demand for player 2 (or 1) but not vice versa.

Now consider a new allocation rule AR-$(\widetilde{eh})$ which gives allocations $\alpha_n^{(\widetilde{eh})}(\vec{X}, \vec{D})$ to player n as follows:

$$\alpha_n^{(\widetilde{eh})}(\vec{X}, \vec{D}) = \alpha_n^{(eh)}(\vec{X}, \vec{D}) + [\alpha_n^d(\vec{X}^{c*}, \vec{D}) - \alpha_n^{(eh)}(\vec{X}^{c*}, \vec{D})] \tag{13}$$

By Theorem 2.1 AR-$(\widetilde{eh})$ is in the core of $\mathbf{SAG}(\vec{D})$ or **AGE** when the inventory decision is $\vec{X}^{c*}$. Thus under the modified allocation rule AR-$(\widetilde{eh})$ we always achieve the first best in the duopoly case. From (13) observe that the allocations under AR-(eh) and AR-$(\widetilde{eh})$ differ by a constant which depends on the first best solution $\vec{X}^{c*}$ and the demand realizations. These constants can be interpreted as "side-payments". Thus we conclude that for a duopoly model there exists a transfer pricing scheme with a schedule of side-payments that gives the first best solution.

We illustrate using an example.

Example 1 *Consider a duopoly system with the following cost structure:* $r_n = 10$, $c_n = 1.2$, $v_n = -1$, $t_{1,2} = 1$, *and* $t_{2,1} = 2$; *that is, transhipment costs are asymmetric. Each player faces a demand which is uniformly distributed between* $[0, 100]$. *The newsvendor solution* $\vec{X}^{nb} = (80, 80)$. *The first best solution* $\vec{X}^{c*} = (77, 62)$. *Notice that* $\vec{X}^{c*} < \vec{X}^{nb}$ *and hence the conditions of Theorem 3.2 are satisfied. Now consider the decentralized system. It can be shown that* $\gamma_1 = 0.6711$ *and* $\gamma_2 = 0.034$ *achieves the first best. The corresponding transfer prices are:* s_{12} = *\$0.34 and* s_{21} = *\$6.04.*

Now consider the same duopoly model but with transhipment costs of $t_{1,2} = 1$ *and* $t_{2,1} = 3$. *The first best solution is* $\vec{X}^{c*} = (85, 54)$. *Thus while* $X_1^{c*} > X_1^{nb}$, $X_2^{c*} < X_2^{nb}$ *and the conditions of Theorem 3.2 are not satisfied. Nevertheless, in the decentralized system, first best is achieved whenever* $\gamma_1 = 1.953$ *and* $\gamma_2 = -0.03$. *As we can see this violates the constraints on* $\gamma_i \in [0, 1]$ *for* $i = 1, 2$ *and hence is not in the core. The corresponding transfer prices are:* s_{12} = *- \$0.3 and* s_{21} = *\$15.624. This is to be interpreted as follows:*

- *If player 1 supplies goods to player 2, then player 1* pays *player 2 a unit price of \$0.3 in addition. Thus player 2's total marginal profit from satisfying residual demand using goods from player 1 equals \$10.3 (\$10 per unit surplus plus \$0.3 payment from player 1).*

- *If player 2 supplies goods to player 1, then player 1* pays *player 2 a unit price of \$7.624 and, in addition, player 2 keeps the marginal surplus of \$8.0 generated from the exchange. Thus player 2's total marginal profit from satisfying residual demand using goods from player 1 equals \$15.624.*

Clearly, such a transfer pricing mechanism is not in the core. However, we can use AR-$\widetilde{eh}$ *such that the corresponding allocations given by (13) is in the core. To construct this, we need to compute the dual prices used in (12). The excess stock and excess demand for each player are given as:*

$$H_1^c = \max\{85 - D_1, 0\}; \quad E_1^c = \max\{D_1 - 85, 0\}$$

$$H_2^c = \max\{54 - D_2, 0\}; \quad E_2^c = \max\{D_2 - 54, 0\}$$

The exchange of goods between the players occur whenever one player has excess demand and the other has excess stock. Based on the demand realizations, this gives rise to four scenarios (denoted by R1,R2,R3, and R4) with the corresponding dual prices: The allocations to player 1

Scenario	Event	Dual Prices			
		ν_1	ν_2	δ_1	δ_2
R1	$0 < H_1^c < E_2^c$	10	–	–	0
R2	$0 < E_2^c < H_1^c$	0	–	–	10
R3	$0 < H_2^c < E_1^c$	–	8	0	–
R4	$0 < E_1^c < H_2^c$	–	0	8	–

Table 1: Dual Prices for $t_{12} = 1$ and $t_{21} = 3$.

under AR-$\widetilde{eh}$ *for each of the regions are as follows:*

$$\alpha_1^{(\widetilde{eh})}(\vec{X}, R1) = -0.3H_1 + 10.3H_1^c$$
$$\alpha_1^{(\widetilde{eh})}(\vec{X}, R2) = -0.3E_2 + 0.3E_2^c$$
$$\alpha_1^{(\widetilde{eh})}(\vec{X}, R3) = -7.624H_2 + 7.624H_2^c$$
$$\alpha_1^{(\widetilde{eh})}(\vec{X}, R4) = -7.624E_1 + 15.624E_1^c$$

Consider the compensation in R1. Earlier (under the simple transfer pricing rule), player 1 paid \$0.3 per unit for supplying goods to player 2. Now under the modified rule, while player 1 pays \$0.3 per unit to player 2 for each unit shipped, player 2 compensates player 1 an amount proportional to player 1's realized demand, decreasing as player 1's realized demand increases. The exact compensation is $10.3H_1^c = 10.3 * (85 - D_1) = 875 - 10.3D_1$. *Other allocations can also be interpreted analogously. Similarly, the allocations to player 2 under AR-*$\widetilde{eh}$ *for each of the regions are as follows:*

$$\alpha_2^{(\widetilde{eh})}(\vec{X}, R1) = 10.3H_1 - 10.3H_1^c$$

$$\alpha_2^{(\widetilde{eh})}(\vec{X}, R2) = 10.3E_2 - 0.3E_2^c$$
$$\alpha_2^{(\widetilde{eh})}(\vec{X}, R3) = 15.624H_2 - 7.624H_2^c$$
$$\alpha_2^{(\widetilde{eh})}(\vec{X}, R4) = 15.624E_1 - 15.624E_1^c$$

The resulting NE is first best and these modified allocations are in core of $\mathbf{SAG}(\vec{D})$ *whenever the inventory vector is first best.*

Alternately, if the players agree on **AGE** *as the solution concept, then the modified allocation mechanism can be interpreted as the original transfer prices plus a fixed fee from player 2 to 1. The fixed fee is given by the expected value of the second term on the right hand side of (13). In the current example, this fixed transfer to player 1 is computed as the expected value of the second term on the right hand sides of the modified allocations to player 1 given by:*

$$\mathbf{E}_{R1}[10.3H_1^c] + \mathbf{E}_{R2}[0.3E_2^c] + \mathbf{E}_{R3}[7.624H_2^c] + \mathbf{E}_{R4}[15.624E_1^c] = \$29.5966$$

Thus, under **AGE**, *a two-part tariff allocation rule achieves the first best.*

Finally, we explore the issue of meta-core which is straightforward for a duopoly model.

Proposition 3.5 *For a duopoly system, all Pareto optimal allocations that are in the core of either* $\mathbf{SAG}(\vec{D})$ *or* **AGE** *are also in the meta–core defined by (4).*

Thus any allocation rule that achieves the first best inventory solution and is in the core of $\mathbf{SAG}(\vec{D})$ or **AGE** is in the meta-core. For example, AR-(eh) is in the meta-core and hence implementable. Clearly, the meta-core is not unique since Corollary 4.2 of ABZ99 gives another general mechanism to achieve first best.

4 Oligopoly Model with Claims

In the previous section on duopoly models, we demonstrated that transfer pricing mechanisms (with or without side-payments) exist that give the first best inventory solution (Theorem 3.2). In general, for oligopolies, this is unlikely to be true. However, if we make some assumptions regarding the cost / revenue and / or demand parameters, we can derive some sharper results. To this extent, we present two oligopoly models. The first one assumes symmetric players. For this model, we study the strategic impact of the intuitive allocation policies (AR-h and AR-e) introduced in the previous section. We illustrate that a suitable randomization of AR-h and AR-e gives the first best solution while a transfer pricing scheme does not. The second oligopoly model we present is a single-warehouse multi-retailer system. This model is similar to the one discussed by Gerchak and Gupta (1991), Robinson (1993), and Hartman and Dror (1996). For the equivalent decentralized system, the pooled inventory is assumed to be claimed. We present a marginal allocation rule which is intuitive in the sense that a player is given a surplus which is a function of his marginal contribution to the coalition. We show that the first best solution can be obtained under this allocation rule. For both models, we develop conditions under which the respective allocation rules considered are in the meta-core.

Before we proceed with the analysis of these models, we need to revisit the solution concepts to be used. Specifically, until now we have used $\mathbf{SAG}(\vec{D})$ as the solution concept for the allocation game. Can we continue to use it or do we need to resort to **AGE**? Recall from Proposition 3.1 that in a duopoly model any non-negative allocation of surplus is in the core of $\mathbf{SAG}(\vec{D})$. This is not true, in general, for oligopoly models. Consider the allocation rules AR-h and AR-e. Clearly, the allocations under these rules are non-negative but they may not be in the core for $\mathbf{SAG}(\vec{D})$ in an oligopoly model. We illustrate this using a counter-example for AR-h.

Example 2 *Consider a symmetric system. That is, all cost parameters are identical. For instance, let* $r_n = 10$ *and* $v_n = 5\ \forall n$ *and* $t_{i,n} = 1\ \forall i, n$. *Then the optimal shipping pattern is irrelevant: Any shipment of excess supply to excess demand generates equal excess profits of* $10 - 5 - 1 = 4$ *per unit. Consider an allocation system where the entire excess profit is allocated to the retailer(s) with excess stocks (i.e., AR-h policy is used). Consider the case when* $H_1 = 5$, $H_2 = 3$, $E_3 = 3$ *and* $E_4 = 1$. *Total excess profits generated is equal to* 16. *Based on the allocation mechanism just described,* $\vec{\alpha} = (10, 6, 0, 0)$. *However, the coalition* $\{1, 3\}$ *can generate a surplus of* 12 *by themselves thus contradicting the core constraints (1a).*

Similarly, we can construct an example to illustrate that AR-e may not be in core of $\mathbf{SAG}(\vec{D})$. Therefore, we use **AGE** as the solution concept for the allocation game for the oligopoly models considered in this paper.

Finally, as in the duopoly model, we will assume here that (i) demands are independent and (ii) *all* customers at retailer n will accept service out of location i if need be; that is, $\beta_{i,n} = 1$ for all $i, n \in \{1, 2\}$.

4.1 Symmetric Players

We assume that all players (i) have identical cost structure with revenue equal to r and salvage value equal to v, (ii) face demands drawn from independent and identical distributions. Furthermore, we assume that the transshipment cost between any two locations is positive but identical and denoted by $t > 0$. We first present a general result regarding the existence of a unique symmetric pure strategy NE. Unfortunately, while we conjecture that the conditions will indeed be satisfied by the payoff functions under AR-h and AR-e, we are unable to show it as yet. However, we show that a symmetric pure strategy NE exists. Furthermore, allocation rules AR-h and AR-e are in the core of **AGE** at this equilirium. Finally, we will also show that a suitable randomization between AR-h and AR-e gives the first best solution while a transfer pricing scheme does not.

The following theorem establishes (i) existence of a symmetric pure strategy NE, and (ii) sufficient conditions for uniqueness of a pure strategy NE for symmetric oligopolies.

Theorem 4.1 *Let $J_i(\vec{X})$ be the twice-differentiable payoff function of player i in an oligopoly with N players. Then there exists a symmetric pure strategy NE. Furthermore, if conditions SO1 and SO2 below hold, then the symmetric equilibrium is also unique.*

SO1:

$$\frac{\partial^2 J_i(\vec{X})}{\partial {X_i}^2} - \frac{\partial^2 J_i(\vec{X})}{\partial X_i \partial X_j} < (\text{ or } >) 0 \qquad \forall X_i, X_j$$

SO2:

$$\frac{\partial^2 J_i(\vec{X})}{\partial {X_i}^2} + (N-1)\frac{\partial^2 J_i(\vec{X})}{\partial X_i \partial X_j} < (\text{ or } >) 0 \qquad \forall X_i, X_j$$

Condition SO1 eliminates all non-symmetric equilibria while condition SO2 eliminates all other symmetric equilibria. The sign of the inequalities in SO1 and SO2 need not be the same. Observe that if the payoff function of player i is concave in X_i and it is sub-modular then condition SO2 is trivially satisfied.

Now consider the allocation rules AR-h and AR-e. Recall that H_i is the excess inventory of player i after satisfying it's own demand and E_i is the excess demand before any transhipments. Define $H = \sum_{i=1}^{N} H_i$ and $E = \sum_{i=1}^{N} E_i$. Clearly, the total transhipments, $T = \min\{H, E\}$ and the total surplus generated is $(r - v - t)T$. Then, under AR-h, player i gets

$$\alpha_i^h(\vec{X}, \vec{D}) = (r - v - t)\frac{H_i}{H}T,$$

and under AR-e it gets

$$\alpha_i^e(\vec{X}, \vec{D}) = (r - v - t)\frac{E_i}{E}T.$$

Unfortunately, we are unable to show if Conditions SO1 and SO2 are satisfied for AR-h and AR-e. Therefore, we cannot claim uniqueness of NE. However, given the existence of a symmetric pure strategy NE, we have the following result.

Proposition 4.1 *For an oligopoly with symmetric players AR-h and AR-e are in the core of* **AGE** *whenever* $X_i = X_j$ *for all* $i, j \in \mathcal{N}$ *and demands are Normally distributed.*

Before we present our two main results for this subsection, we need to introduce some further notation. Define $H_{-i} = \sum_{j \neq i} H_i$, and $E_{-i} = \sum_{j \neq i} E_i$. Let H_i' and E_i' represent the first derivative of H_i and E_i respectively w.r.t. X_i. Let H_i^c, H_{-i}^c, and $(H_i^c)'$ respectively be the values of H_i, H_{-i}^c, and H_i' evaluated at the first best solution ($\vec{X}^{c*}$); similarly define E_i^c, E_{-i}^c and $(E_i^c)'$.

Then,

Proposition 4.2 *The total inventory in the decentralized system under AR-h (respectively, AR-e) is above (respectively, below) the total inventory in the first best solution.*

We use a sample path approach for the proof. It follows by showing that for every sample path the first partial of the payoff function of player i weakly dominates (respectively, dominated by) the first partial of the centralized profit function with respect to X_i under AR-h (respectively, AR-e) for all i.

In Proposition 4.2, observe that the symmetry of the equilibrium also implies that there is component-wise dominance between equilibrium inventory positions in AR-h (or AR-e) and the first best. Thus strategically players over-invest in inventory under AR-h and under-invest under AR-e. Given this bahavior, it may then seem plausible that a suitable combination of these two rules should give the first best solution. This is indeed true.

Theorem 4.2 *Assume that AR-h and AR-e are in the core of* **AGE**. *Then there exists a constant* $\gamma \in [0, 1]$ *where*

$$\gamma = \frac{\mathbf{E}_{H^c \leq E^c}\left(\frac{(E_i^c)' H^c E_{-i}^c}{(E^c)^2} - (H_i^c)'\right)}{\mathbf{E}_{H^c \leq E^c}\left(\frac{(E_i^c)' H^c E_{-i}^c}{(E^c)^2} - (H_i^c)'\right) - \mathbf{E}_{H^c > E^c}\left(\frac{(H_i^c)' E^c H_{-i}^c}{(H^c)^2} - (E_i^c)'\right)}$$

such that an allocation policy that allocates based on AR-h with probability γ *and based on AR-e with probability* $1-\gamma$ *achieves the first best solution. Furthermore, this policy is in the meta-core if demands are normally distributed.*

We make the following observations:

- The randomization between AR-h and AR-e is similar to the one used in Theorem 3.2 for the duopoly model. Unlike the case of duopoly where we had to impose conditions on when the first best solution is achieved (see Theorem 3.2), symmetry in this oligopoly model gives us the first best. Furthermore, unlike in the duopoly case, randomization for this example does not uniformly have a transfer pricing interpretation. It does but only under certain realizations of demand. For example, if total excess stock (H) is smaller than the total excess demand (E), then all players with excess stock get a fixed marginal price for each unit they tranship. Similarly, if total excess stock (H) is larger than the total excess demand (E), then all players with excess demand get a fixed marginal price for each unit they receive.
- Finally, observe that the Theorem only requires that AR-h and AR-e be in the core of **AGE** but does not directly require that demands be Normally distributed. The Normal distribution assumption is required only to show membership in the meta-core.

4.2 A Single Warehouse Multi-Retailer System

Consider a centralized system with a single central warehouse that satisfies demands of multiple retailers. Furthermore, assume that the transhipment cost between the central warehouse and the retailers is zero. The procurement cost at the central warehouse is c and salvage value of leftover goods is v. The revenues (r_i) and the demand distributions that each retailer faces are independent but non-identical. Now consider an equivalent decentralized system with claims; that is, all inventory in the central warehouse is claimed. Assume all retailers face identical procurement costs ($c_i = c$) and salvage values ($v_i = v$).

Observe that there is no local inventory; that is $\vec{X} = \vec{0}$. Since there is only one warehouse, we drop the index for the warehouse in the relevant variables. So we write the claimed inventories as $\vec{Y} = (Y_1, \ldots, Y_N)$ and the transhipment from the central warehouse to retailer $n \in \mathcal{N}$ is denoted by q_n with $\vec{Q} = (q_1, \ldots, q_N)$. Let $W^*_{\mathcal{S}}(\vec{Y}, \vec{D})$ be the value of the coalition $\mathcal{S} \subseteq \mathcal{N}$ expressed as follows:

$$W^*_{\mathcal{S}}(\vec{Y},\vec{D}) = \max_{\vec{Q}} \sum_{n\in\mathcal{S}} [(r_n - v)q_n] \tag{14a}$$

subject to:

$$\sum_{n\in\mathcal{S}} q_n \le \sum_{n\in\mathcal{S}} Y_n \tag{14b}$$

$$0 \le q_n \le D_n \quad \forall n \in \mathcal{S} \tag{14c}$$

Let $W^*_{\mathcal{S}}(\vec{Y}) = \mathbf{E}_{\vec{D}} W^*_{\mathcal{S}}(\vec{Y},\vec{D})$. Define a marginal allocation rule as follows:

Incremental Contribution Rule (AR-ic) Rank order the N players according to their revenues such that $r_1 \le r_2 \le \cdots \le r_N$. Then player n is allocated a profit α_n given by

$$\alpha^{ic}_n(\vec{Y}) = W^*_{\mathcal{S}_n\cup\{n\}}(\vec{Y}) - W^*_{\mathcal{S}_n}(\vec{Y})$$

where $\mathcal{S}_n = \{1,...,n-1\}$.

Observer that AR-ic gives each player what it contributes to the coalition. Furthermore, $\alpha^{ic}_n \ge 0$ since the function $W^*_{\mathcal{S}}$ is super-additive.

Proposition 4.3 *Consider the single warehouse multiple retailer system with claims. Suppose the retailers face identical procurement costs and salvage values and transhipments costs are zero but sell at different prices and face independent, non-identical demands. If AR-ic is in the core of* **AGE***, then the inventory decision in the decentralized system is first best.*

It is still unclear, however, if allocations resulting from AR-ic are in the core of **AGE**. If they are, then AR-ic becomes a candidate for implementation. If not then we can use Theorem 2.1 to suitably modify AR-ic to belong to the core of **AGE**.

In general, it is very hard to show if a certain allocation mechanism is in the meta-core; that is, satisfy the conditions (4). We can make more precise statements about membership in the core of **AGE** and the meta-core under further assumptions.

Proposition 4.4 *Consider the single warehouse multiple retailer system with claims. Suppose the retailers face identical procurement costs and salvage values and transhipments costs are zero. In addition, assume that the retailers face identical revenues and that the demand distributions are Normal with identical standard deviations but non-identical means. Then* *allocations under AR-ic is in the core of* **AGE**. *Furthermore, AR-ic belongs to the meta-core defined by (4).*

In this model, our context of inventory is incidental. The same model can capture, for example, a situation when there is a single factory serving multiple product managers each responsible for a product line. In such settings the key decisions include allocation of capacity to each product manager and investment in total capacity. One option is to have a plant manager make these decisions. Alternately, using the concept of claims, we can allow each product manager to report their own capacity choices which are then pooled together in a single facility. Then Proposition 4.3 the allocation rule AR-ic (perhaps modified) ensures that the total capacity investment will be first best.

5 Summary and Conclusion

In this paper we use the framework developed by Anupindi, Bassok, and Zemel (1999) to analyze a general duopoly model and several oligopoly models of distribution systems with search. There are two distinctive features of the systems we study: the players (i) *ex ante* behave non-cooperatively on the inventory deployment decision and (ii) *ex post* behave cooperatively in the exchange of goods between locations to match excess stock with excess demand. Solution concepts include the notion of core for the ex post decision and the notion of Nash Equilibrium for the ex ante decision. Common inventories in central warehouses are "claimed" by each player.

For a duopoly model, we first develop sufficient conditions for the uniqueness of the Nash Equilibrium. We then propose two intuitive allocation rules for the duopoly model without claimed inventory. The first rule allocates all surplus to a retailer who satisfies the demand while the second rule allocates it to the retailer who generates the demand. We show that the Nash Equilibrium under these allocation mechanisms is unique. We then show that the retailer respectively over- and under-invest in inventories under these allocation rules. We then demonstrate that a suitable modification of these two rules gives the transfer pricing mechanism. Subsequently, we derive necessary and sufficient conditions when a transfer pricing mechanism results in the first best solution.

We then present two oligopoly models. For an oligopoly with symmetric players, we first develop general conditions for the uniqueness of a symmetric Nash Equilibrium. We then show that the intuitive allocation rules used in the duopoly model belong to the core for the allocation game in expectation. We also show that a randomization of these rules gives the first best solution. Unlike in the duopoly setting, this modification, however, does not have a transfer pricing interpretation. Using the concept of claims, we then study a single warehouse multi-retailer system. We suggest an allocation rule which allocates surplus profit to a player based on its incremental contribution to a coalition. We show that this rule gives the first best solution.

There are several ways in which the work here can be extended. Clearly, it will be useful to explore whether there exist “simple” mechanisms that give the first best solution in oligopoly models other than those considered here. Like in Anupindi and Bassok (1999), it will be interesting to consider the vertical interaction between a manufacturer and the retailers but under retailer-driven search. Extending the general framework in Anupindi, Bassok, and Zemel (1999) and its application to models considered in this paper to incorporate asymmetric information is another area of future research.

A Appendix

Proof of Proposition 3.3:

For the decentralized system, we first consider AR-h. Define

$$L_i(X_i) = r_i\mathbf{E}[\min\{D_i, X_i\}] + v_i\mathbf{E}[\max\{X_i - D_i, 0\}].$$

Then the profit function of player 1 under AR-h is written as:

$$J_1^h(X_1,X_2) = -c_1X_1 + L_1(X_1) + (r_2 - v_1 - t_{1,2})\int_0^{X_1}\int_{X_2}^{X_1+X_2-d_1}(d_2 - X_2)dF_2(d_2)dF_1(d_1)$$
$$+(r_2 - v_1 - t_{1,2})\int_0^{X_1}\int_{X_1+X_2-d_1}^{\infty}(X_1 - d_1)dF_2(d_2)dF_1(d_1) \quad (15)$$

Similarly, the expected payoff function of player 2 is:

$$J_2^h(X_1,X_2) = -c_2X_2 + L_2(X_2) + (r_1 - v_2 - t_{2,1})\int_0^{X_2}\int_{X_1}^{X_1+X_2-d_2}(d_1 - X_1)dF_1(d_1)dF_2(d_2)$$
$$(r_1 - v_2 - t_{2,1})\int_0^{X_2}\int_{X_1+X_2-d_2}^{\infty}(X_2 - d_2)dF_1(d_1)dF_2(d_2) \quad (16)$$

We write the first partials of profit functions for players 1 and 2 as follows:

$$\frac{\partial J_1^h(X_1,X_2)}{\partial X_1} = -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) + (r_2 - v_1 - t_{1,2})\int_0^{X_1}\bar{F}_2(X_1 + X_2 - d_1)dF_1(d_1)$$

and,

$$\frac{\partial J_2^h(X_1,X_2)}{\partial X_2} = -c_2 + r_2\bar{F}_2(X_2) + v_2F_2(X_2) + (r_1 - v_2 - t_{2,1})\int_0^{X_2}\bar{F}_1(X_1 + X_2 - d_2)dF_2(d_2)$$

It is straightforward to see that

$$\frac{\partial J_i^{nb}}{\partial X_i} \le \frac{\partial J_i^h}{\partial X_i}\forall i = 1,2.$$

This implies that $\vec{X}^{nb} \le \vec{X}^{h*}$.

Now consider AR-e. The expected payoff function of player 1 is given by:

$$J_1^e(X_1,X_2) = -c_1X_1 + L_1(X_1) + (r_1 - v_2 - t_{21})\int_0^{X_2}\int_{X_1}^{X_1+X_2-d_2}(d_1 - X_1)dF_1(d_1)dF_2(d_2)$$
$$+(r_1 - v_2 - t_{21})\int_0^{X_2}\int_{X_1+X_2-d_2}^{\infty}(X_2 - d_2)dF_1(d_1)dF_2(d_2) \quad (17)$$

and,

$$J_2^e(X_1,X_2) = -c_2X_2 + L_2(X_2) + (r_2 - v_1 - t_{12})\int_0^{X_1}\int_{X_2}^{X_1+X_2-d_1}(d_2 - X_2)dF_2(d_2)dF_1(d_1)$$
$$+(r_2 - v_1 - t_{12})\int_0^{X_1}\int_{X_1+X_2-d_1}^{\infty}(X_1 - d_1)dF_2(d_2)dF_1(d_1) \quad (18)$$

Differentiating, we write

$$\frac{\partial J_1^e(X_1,X_2)}{\partial X_1} = -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) - (r_1 - v_2 - t_{21})\int_0^{X_2}[F_1(X_1+X_2-d_1) - F_1(X_1)]dF_2(d_2)$$

and,

$$\frac{\partial J_1^e(X_1,X_2)}{\partial X_2} = -c + r_2\bar{F}_2(X_2) + vF_2(X_2) - (r_2 - v_1 - t_{12})\int_0^{X_1}[F_2(X_1+X_2-d_1) - F_2(X_2)]dF_1(d_1)$$

It is straightforward to see that

$$\frac{\partial J_i^{nb}}{\partial X_i} \geq \frac{\partial J_i^e}{\partial X_i}\forall i = 1,2.$$

This implies that $\vec{X}^{nb} \geq \vec{X}^{e*}$. ∎

Proof of Proposition 3.4:

Part(1): When transhipment costs are zero and the cost and revenue parameters are identical, the centralized system keeps inventory at only one location. Denote this inventory by X^{c*}. Then,

$$F_c(X^{c*}) = \frac{r-c}{r-v},$$

where $F_c(\cdot)$ is the cumulative demand distribution. It is well known that $X^{c*} \leq X_1^{nb} + X_2^{nb}$. From Proposition 3.3, this implies that $X^{c*} \leq X_1^{h*} + X_2^{h*}$. Now consider AR-e. Since $\vec{X}^{e*}$ is the Nash Equilibrium under AR-e, we have

$$\frac{\partial J_1^e}{\partial X_1}\Big|_{\vec{X}^{e*}} = (r-c) - (r-v)\left[F_1(X_1^{e*})\bar{F}_2(X_2^{e*}) + \int_0^{X_2^{e*}} F_1(X_1^{e*} + X_2^{e*} - D_2)dF_2(D_2)\right]$$

which can be re–written as

$$\begin{aligned}
F_c(X^{c*}) &= F_1(X_1^{e*})\bar{F}_2(X_2^{e*}) + \int_0^{X_2^{e*}} F_1(X_1^{e*} + X_2^{e*} - D_2)dF_2(D_2)\\
&= \Pr\{(D_1 \leq X_1^{e*}) \wedge (D_2 \geq X_2^{e*})\} + \Pr\{(D_2 \leq X_2^{e*}) \wedge (D_1 + D_2 \leq X_1^{e*} + X_e^{2*})\}\\
&= \Pr\{D_1 + D_2 \leq X_1^{e*} + X_2^{e*}\} + \Pr\{(D_1 \leq X_1^{e*}) \wedge (D_2 \geq X_2^{e*}) \wedge (D_1 + D_2 \geq X_e^{1*} + X_2^{e*})\}\\
&\geq \Pr\{D_1 + D_2 \leq X_1^{e*} + X_2^{e*}\}\\
&= F_c(X_1^{e*} + X_2^{e*}).
\end{aligned}$$

The result follows from the monotonicity of $F_c(\cdot)$.

Part (ii): When all costs, revenue, and demand parameters are identical, the Nash equilibrium is also symmetric. We need to first derive the first partials of the profit function in the centralized system. We specialize the expressions (19)–(20) derived later for the more general case, to give:

$$\begin{aligned}
\frac{\partial J^c(X_1,X_2)}{\partial X_1} &= -c + r\bar{F}_1(X_1) + vF_1(X_1) + (r-v-t)\int_0^{X_1}\bar{F}_2(X_1+X_2-d_1)dF_1(d_1)\\
&\quad -(r-v-t)\int_0^{X_2}[F_1(X_1+X_2-d_2) - F_1(X_1)]dF_2(d_2)
\end{aligned}$$

and,

$$\frac{\partial J^c(X_1,X_2)}{\partial X_2} = -c + r\bar{F}_2(X_2) + vF_2(X_2) + (r-v-t)\int_0^{X_2} \bar{F}_1(X_1+X_2-d_2)dF_2(d_2)$$
$$-(r-v-t)\int_0^{X_1}[F_2(X_1+X_2-d_1) - F_2(X_2)]dF_1(d_1)$$

It is easy to see that

$$\frac{\partial J_i^e}{\partial X_i} \leq \frac{\partial J^c}{\partial X_i} \leq \frac{\partial J_i^h}{\partial X_i} \forall i = 1,2.$$

Symmetry of the Nash equilibrium implies that $\vec{X}^{e*} \leq \vec{X}^{c*} \leq \vec{X}^{h*}$. ∎

Proof of Theorem 3.2:

First we develop the profit function for the centrally coordinated system. The expected profits of the central system for stock levels X_1 and X_2 is:

$$J^c(X_1,X_2) = -c_1X_1 + r_1\mathbf{E}[\min\{D_1,X_1\}] + v_1\mathbf{E}[\max\{X_1-D_1,0\}]$$
$$+(r_2-v_1-t_{12})\mathbf{E}[\max\{0,\min\{D_2-X_2,X_1-D_1\}\}]$$
$$-c_2X_2 + r_2\mathbf{E}[\min\{D_2,X_2\}] + v_2\mathbf{E}[\max\{X_2-D_2,0\}]$$
$$+(r_1-v_2-t_{21})\mathbf{E}[\max\{0,\min\{D_1-X_1,X_2-D_2\}\}]$$

Differentiating with respect to X_1 and X_2 respectively, we obtain:

$$\frac{\partial J^c(X_1,X_2)}{\partial X_1} = -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) + (r_2-v_1-t_{12})\int_0^{X_1} \bar{F}_2(X_1+X_2-d_1)dF_1(d_1)$$
$$-(r_1-v_2-t_{21})\int_0^{X_2}[F_1(X_1+X_2-d_2) - F_1(X_1)]dF_2(d_2) \quad (19)$$

and,

$$\frac{\partial J^c(X_1,X_2)}{\partial X_2} = -c_2 + r_2\bar{F}_2(X_2) + v_2F_2(X_2) + (r_1-v_2-t_{21})\int_0^{X_2} \bar{F}_1(X_1+X_2-d_2)dF_2(d_2)$$
$$-(r_2-v_1-t_{12})\int_0^{X_1}[F_2(X_1+X_2-d_1) - F_2(X_2)]dF_1(d_1) \quad (20)$$

Now consider the AR-(eh) policy. It implies that if $E_1 > 0$ and $H_2 > 0$, then a fraction $\gamma_1 \in [0,1]$ of the surplus profits go to player 2 and the remaining to player 1. If $H_1 > 0$ and $E_2 > 0$ then a fraction $\gamma_2 \in [0,1]$ of the surplus profits go to player 1 and the remaining to player 2. Then,

$$J_1^{(eh)}(X_1,X_2) = -c_1X_1 + r_1\mathbf{E}[\min\{D_1,X_1\}] + v_1\mathbf{E}[\max\{X_1-D_1,0\}]$$
$$+\gamma_2(r_2-v_1-t_{12})\mathbf{E}[\max\{0,\min\{D_2-X_2,X_1-D_1\}\}]$$
$$+(1-\gamma_1)(r_1-v_2-t_{21})\mathbf{E}[\max\{0,\min\{D_1-X_1,X_2-D_2\}\}]$$

and,

$$J_2^{(eh)}(X_1,X_2) = -c_2X_2 + r_2\mathbf{E}[\min\{D_2,X_2\}] + v_2\mathbf{E}[\max\{X_2-D_2,0\}]$$
$$+(1-\gamma_2)(r_2-v_1-t_{12})\mathbf{E}[\max\{0,\min\{D_2-X_2,X_1-D_1\}\}]$$
$$+\gamma_1(r_1-v_2-t_{21})\mathbf{E}[\max\{0,\min\{D_1-X_1,X_2-D_2\}\}]$$

Using the functions $A_i(X_1, X_2)$ and $B_i(X_1, X_2)$, we simplify to write

$$\begin{aligned}\frac{\partial J_1^{(eh)}}{\partial X_1} &= -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) - (1-\gamma_1)A_1(X_1,X_2) + \gamma_2 B_1(X_1,X_2) \\ &= -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) - A_1(X_1,X_2) + B_1(X_1,X_2) \\ &\quad +[\gamma_1 A_1(X_1,X_2) - (1-\gamma_2)B_1(X_1,X_2)] \\ &= \frac{\partial J^c}{\partial X_1} + [\gamma_1 A_1(X_1,X_2) - (1-\gamma_2)B_1(X_1,X_2)] \end{aligned} \tag{21}$$

Similarly, it can be shown that,

$$\frac{\partial J_2^{(eh)}}{\partial X_2} = \frac{\partial J^c}{\partial X_2} + [\gamma_2 B_2(X_1,X_2) - (1-\gamma_1)A_2(X_1,X_2)] \tag{22}$$

If we can find $\gamma_1 \in [0,1]$ and $\gamma_2 \in [0,1]$ such that the second term in the RHS of (21) and (22) evaluated at (X_1^{c*}, X_2^{c*}) are zero, then it is clear that the Nash equilibrium solution of this AR-(eh) policy is the same as the centralized solution, component wise. Recall that $A_i^c = A_i(X_1^{c*}, X_2^{c*})$ and $B_i^c = B_i(X_1^{c*}, X_2^{c*})$ for $i = 1, 2$. So we need to find γ_1 and γ_2 such that

$$\begin{aligned}\gamma_1 A_1^c - (1-\gamma_2)B_1^c &= 0 \\ \gamma_2 B_2^c - (1-\gamma_1)A_2^c &= 0\end{aligned}$$

Solving these simultaneous equations, we get that,

$$\begin{aligned}\gamma_1 &= \frac{(B_2^c - A_2^c)B_1^c}{B_2^c A_1^c - B_1^c A_2^c} \\ \gamma_2 &= \frac{(A_1^c - B_1^c)A_2^c}{B_2^c A_1^c - B_1^c A_2^c}\end{aligned}$$

We still need to check if $\gamma_i \in [0,1]$. Using the definitions of $A_i(X_1, X_2)$ and $B_i(X_1, X_2)$, we rewrite,

$$\frac{\partial J^c(X_1,X_2)}{\partial X_1} = -c_1 + r_1\bar{F}_1(X_1) + v_1F_1(X_1) + B_1(X_1,X_2) - A_1(X_1,X_2) \tag{23}$$

$$\frac{\partial J^c(X_1,X_2)}{\partial X_2} = -c_2 + r_2\bar{F}_2(X_2) + v_2F_2(X_2) + A_2(X_1,X_2) - B_2(X_1,X_2) \tag{24}$$

Observe that if there were no transhipments between the two players, each will play independently as a newsvendor problem and the optimal inventory decisions, denoted by (X_1^{nb}, X_2^{nb}) will satisfy the following:

$$F_i(X_i^{nb}) = \frac{r_i - c_i}{r_i - v_i} \qquad \text{for } i = 1, 2.$$

Comparing the independent newsvendor solution to the optimal centralized solution, we have the following three cases to consider:

- $X_i^{c*} < X_i^{nb}$ for $i = 1, 2$: Evaluating (23) at (X_1^{c*}, X_2^{c*}) we get that

$$0 = -c_1 + r_1\bar{F}_1(X_1^{c*}) + v_1F_1(X_1^{c*}) + B_1^c - A_1^c$$

  But from monotonicity of $F_i(\cdot)$, we have

$$F_i(X_i^{c*}) < \frac{r_i - c_i}{r_i - v_i}$$

or,

$$-c_1 + r_1 \bar{F}_1(X_1^{c*}) + v_1 F_1(X_1^{c*}) > 0$$

which implies that $B_1^c < A_1^c$. Similarly, it is seen that $A_2^c < B_2^c$. Substituting these inequalities into (7) and (8), we see that $\gamma_i \in [0,1]$.

- $X_1^{c*} > X_1^{nb}$ and $X_2^{c*} < X_2^{nb}$: Proceeding as before, we derive that $A_1^c < B_1^c$ and $A_2^c < B_2^c$. But this implies that either $\gamma_1 < 0$ or $\gamma_2 < 0$.
- $X_1^{c*} < X_1^{nb}$ and $X_2^{c*} > X_2^{nb}$: Proceeding as before, we derive that $A_1^c > B_1^c$ and $A_2^c > B_2^c$. But this implies that either $\gamma_1 < 0$ or $\gamma_2 < 0$.
- $X_i^{c*} > X_i^{nb}$ for $i = 1, 2$: Proceeding as before it can be shown that $B_1^c > A_1^c$ and $A_2^c > B_2^c$. This implies that $\gamma_i \in [0,1]$.
- $X_i^{c*} = X_i^{nb}$ for at least one i: Without loss of generality assume that $X_1^{c*} = X_1^{nb}$. Then $A_1^c = B_1^c$ which implies that $\gamma_2 = 0$ and $\gamma_1 = 1$. If both $X_i^{c*} = X_i^{nb}$ then any $\gamma_1, \gamma_2 \in [0,1]$ such that $\gamma_1 + \gamma_2 = 1$ will lead to the first best solution.

Therefore, we can achieve the centralized solution using the AR-(eh) policy *if only if* $X_i^{c*} \leq X_i^{nb}$ or $X_i^{c*} \geq X_i^{nb}$ for $i = 1, 2$. ∎

Proof of Theorem 4.1: We will first show existence of a pure strategy NE. Define

$$g(h) = \frac{\partial J_1(h, \ldots, h)}{\partial X_1}.$$

Notice that $g(0) > 0$ and $g(h) < 0$ for suficiently large h. From the continuity of $g(\cdot)$ there exists an h^* such that $g(h^*) = 0$; that is $\frac{\partial J_1(h^*, \ldots, h^*)}{\partial X_1} = 0$. From the symmetry of players, we get that $\frac{\partial J_i(h^*, \ldots, h^*)}{\partial X_i} = 0$ for every i. Hence $(h^*, \ldots, h^*)$ is a pure strategy symmtric NE.

We now prove uniqueness by contradiction. We first eliminate the existence of non-symmetric equilibria. Suppose there are two non-identical equilibria. Let $\vec{X}^*$ be a Nash equilbrium such that $X_1^* < X_2^*$ (without loss of generality). Due of the symmetry we get that $\vec{W}^* \equiv (X_2^*, X_1^*, \ldots, X_N^*)$ is also a NE; observe that $\vec{W}^*$ is obtained by swapping X_1^* with X_2^*. We know that

$$\frac{\partial J_1(\vec{X})}{\partial X_1}\Big|_{\vec{X}^*} = \frac{\partial J_1(\vec{X})}{\partial X_1}\Big|_{\vec{W}^*} = 0.$$

Let

$$g(h) = \frac{\partial J_1(X_1 + h, X_2 - h, X_3^*, \ldots, X_N^*)}{\partial X_1}.$$

Notice that $g(0) = g(X_2^* - X_1^*) = 0$. Suppose that condition SO1 holds. This implies that

$$g'(h) = \frac{\partial^2 J_1(\vec{X})}{\partial {X_1}^2} - \frac{\partial^2 J_1(\vec{X})}{\partial X_2 \partial X_1} > 0.$$

This implies $g(\cdot)$ is an increasing function and hence $g(0) \neq g(X_2^* - X_1^*)$. This gives a contradiction. Similarly, we will get a contradiction if $g(\cdot)$ is a decreasing function. Thus if condition SO1 is satisfied there is no non-symmetrical equilibrium.

Now, suppose there are two symmetric equilbria. Let $\vec{X}^*$ be a Nash equilbrium and $X_i = k$ for every i. Let $\vec{W}^*$ be another symmetric NE. with $W_i^* = m > k$. Clearly $\frac{\partial J_1(\vec{X})}{\partial X_1}\Big|_{\vec{X}^*} = \frac{\partial J_1(\vec{X})}{\partial X_1}\Big|_{\vec{W}^*} = 0$. Let $g(h) = \frac{\partial J_1(k+h,k+h,\ldots k+h)}{\partial X_1}$. Notice that $g(0) = g(m-k) = 0$. Now suppose condition SO2 is satisfied. Then

$$
\begin{aligned}
g'(h) &= \frac{\partial J_1(k+h,\ldots,k+h)}{\partial X_1} + \sum_{j=2}^{N} \frac{\partial^2 J_1(k+h,\ldots,k+h)}{\partial X_j \partial X_1} \\
&= J_1(k+h,\ldots,k+h)X_1 + (N-1)\frac{\partial^2 J_i(k+h,\ldots,k+h)}{\partial X_j \partial X_1} > 0
\end{aligned}
$$

Thus, $g(\cdot)$ is an increasing function and hence $g(0) < g(m-k)$. This leads to a contradiction that $g(m-k) = 0$. Similarly, we will get a contradiction if $g(\cdot)$ is a decreasing function. Thus if condition SO2 is satisfied there is a unique symetric equilibrium. ∎

Proof of Proposition 4.1:

Consider AR-h. The expected profit of player i is:

$$\mathbf{E}\{\frac{H_i}{H}[(r-v-t)T]\}.$$

Using the fact that the base stocks and demand distributions are identical we get that the expected profit of the players are identical. Thus, the expected additional profit of each of the player is:

$$\frac{1}{N}\mathbf{E}\{[(r-v-t)T]\} \equiv \beta \text{ (say).}$$

To show that AR-h is in the core of **AGE** we need to show that for every sub-coalition $\mathcal{S}$, $V_{\mathcal{S}} \leq |\mathcal{S}|\beta$, where $|\mathcal{S}|$ is the number of players in the sub-coalition $\mathcal{S}$. The value of $V_{\mathcal{S}}$ can be written in the following way:

$$V_{\mathcal{S}} = \mathbf{E}\{[(r-v-t)\min\{\sum_{i=1}^{|\mathcal{S}|} H_i, \sum_{i=1}^{|\mathcal{S}|} E_i\}]\}.$$

Thus we need to show that for every sub-coalition $\mathcal{S}$ we get

$$
\begin{aligned}
V_{\mathcal{S}} &= \mathbf{E}\{[(r-v-t)\min\{\sum_{i=1}^{|\mathcal{S}|} H_i, \sum_{i=1}^{|\mathcal{S}|} E_i\}]\} \\
&\leq \frac{|\mathcal{S}|}{N}\mathbf{E}\{[(r-v-t)\min\{\sum_{i=1}^{N} H_i, \sum_{i=1}^{N} H_i\}]\}
\end{aligned}
$$

Alternately, we need to show that,

$$
\begin{aligned}
&\frac{\mathbf{E}\{[(r-v-t)\min\{\sum_{i=1}^{|\mathcal{S}|} H_i, \sum_{i=1}^{|\mathcal{S}|} E_i\}]\}}{|\mathcal{S}|} \\
&\qquad \leq \frac{\mathbf{E}\{[(r-v-t)\min\{\sum_{i=1}^{N} H_i, \sum_{i=1}^{N} E_i\}]\}}{N}
\end{aligned}
$$

We will now prove that the above inequality always holds when demands are Normally distributed. Let X be the inventory level of a player. Then for each player we can easily calculate the expected excess stock after satisfying own demand as $\int_{d=0}^{X}(X-d)dF(d)$. Clearly the sum of the expected excess stock in the system before any transhipments is: $N\int_{d=0}^{X}(X-d)dF(d)$.

We can also calculate the expected inventory in the system after all transhipmens when each player has initially X units of inventory. This is given by $\int_{d=0}^{NX}(NX-d)dF_{1,N}(d)$. The expected number of transhipments, $\mathbf{E}(T)$, is:

$$\mathbf{E}(T) = N\int_{d=0}^{X}(X-d)dF(d) - \int_{d=0}^{NX}(NX-d)dF_{1,N}(d),$$

where $F_{1,N}(\cdot)$ is the cumulative density function of the sum of demands $D_1+\cdots+D_N$. Thus the total surplus profit generated through transhipments is equal to $(r-v-t)\mathbf{E}(T)$. We will calculate the value of $\mathbf{E}(T)$ as a function of the number of the players. Suppose demands are normally distributed as $N(\mu,\sigma)$. It is then easily shown that

$$\int_{d=0}^{X}(X-d)dF(d) = (X-\mu)\Phi(X) + \sigma\phi(\frac{X-\mu}{\sigma}).$$

Let $X=\mu+k\sigma$. Substituting, we get

$$N\int_{d=0}^{X}(X-d)dF(d) = N(k\sigma\Phi(k)+\sigma\phi(k)).$$

Similarly,

$$\int_{d=0}^{NX}(NX-d)dF_{1,N}(d) = \sqrt{N}(k\sigma\Phi(k)+\sigma\phi(k)).$$

We then have,

$$\mathbf{E}(T) = (N-\sqrt{N})(k\sigma\Phi(k)+\sigma\phi(k)) \tag{25}$$

Clearly for every $\mathcal{S}$ we get,

$$\begin{aligned}\frac{(N-\sqrt{N})(k\sigma\Phi(k)+\sigma\phi(k))}{N} &= (1-\frac{1}{\sqrt{N}})(k\sigma\Phi(k)+\sigma\phi(k)) \\ &> (1-\frac{1}{\sqrt{|\mathcal{S}|}})(k\sigma\Phi(k)+\sigma\phi(k)).\end{aligned}$$

This proves that AR-h is in the core. The proof for AR–e is analogous. ∎

Proof of Proposition 4.2 We will first show that $\frac{\partial J_i^h}{\partial X_i} \geq \frac{\partial J_i^c}{\partial X_i}$ and $\frac{\partial J_i^e}{\partial X_i} \leq \frac{\partial J_i^c}{\partial X_i}$ for $i=1,2$ (to show this we use a sample path approach). Since the Nash equilibrium is symmetric, this implies that $\vec{X}^h \geq \vec{X}^c$ and $\vec{X}^e \leq \vec{X}^c$.

Recall that $H_i = \max\{X_i - D_i, 0\}$. Denoting $I_i = X_i - D_i$, we can write $H_i = \max\{I_i, 0\}$ and $E_i = -\min\{I_i, 0\}$. The derivative of I_i w.r.t. to X_i, denoted by $I_i' = 1$. Let $\mathbf{1}\{\mathcal{E}\}$ be the indicator function which takes the value 1 whenever event $\mathcal{E}$ is true and 0 otherwise. Then clearly $H_i' = \mathbf{1}\{I_i > 0\}(I_i)' \geq 0$. Similarly, $E_i' = -\mathbf{1}\{I_i < 0\}I_i' \leq 0$.

In the centralized system, the total profit function for any sample path can be written as:

$$U^c(\vec{X}, \vec{D}) = \sum_{i=1}^{N} DP_i(X_i) + (r - v - t)T.$$

The total expected profit in the centralized system is then $J^c(\vec{X}) = \mathbf{E}U^c(\vec{X})$.

We now derive the first partials of the centralized profit function based on sample paths. We have:

$$\frac{\partial U^c}{\partial X_i} = \frac{\partial DP_i}{\partial X_i} + (r - v - t)\frac{\partial T}{\partial X_i}.$$

We need to consider two cases:

- $H < E$: This implies $T = H = H_i + H_{-i}$. Then,

$$\frac{\partial T}{\partial X_i} = H_i' \geq 0. \tag{26}$$

- $H > E$: This implies $T = E = E_i + E_{-i}$. Then,

$$\frac{\partial T}{\partial X_i} = E_i' \leq 0. \tag{27}$$

Now consider AR-h which asserts the player i will get $\frac{H_i}{H}$ of the total profit. The total profit is $(r - v - t)T$. Thus the expected profit of player i is: $\mathbf{E}\{\frac{H_i}{H}[(r - v - t)T]\}$. That is, the expected indirect profit of player i is

$$\mathbf{E}[IP_i^h(\vec{X})] = (r - v - t)\mathbf{E}\{\frac{H_i}{H}T\}.$$

We need to an expression for $\frac{\partial \mathbf{E}[IP_i^h]}{\partial X_i}$. Let us proceed on a sample path basis. First, Then,

$$\frac{\partial IP_i^h}{\partial X_i} = (r - v - t)\frac{\partial}{\partial X_i}\left[\frac{H_i}{H_i + H_{-i}} \min\{H_i + H_{-i}, E\}\right]$$

Consider a sample path $H < E$. Then,

$$\frac{\partial IP_i^h}{\partial X_i} = (r - v - t)\frac{\partial}{\partial X_i}\left[\frac{H_i}{H_i + H_{-i}}(H_i + H_{-i})\right] = (r - v - t)H_i' \geq 0$$

Comparing with (26), we see that $\frac{\partial IP_i^h}{\partial X_i} = (r - v - t)\frac{\partial T}{\partial X_i}$.

Now suppose $H > E$. Then,

$$\begin{aligned}
\frac{\partial IP_i^h}{\partial X_i} &= (r - v - t)\frac{\partial}{\partial X_i}\left[\frac{H_i}{H_i + H_{-i}}(E)\right] \\
&= (r - v - t)\left[\frac{H_i'E}{H_i + H_{-i}} + \frac{H_iE'}{H_i + H_{-i}} - \frac{H_iEH_i'}{(H_i + H_{-i})^2}\right] \\
&= (r - v - t)\frac{H_i'}{H_i + H_{-i}}\frac{(EH_i + EH_{-i} - EH_i)}{H_i + H_{-i}} \\
&= (r - v - t)\frac{H_i'EH_{-i}}{H^2} \geq 0
\end{aligned}$$

The third step follows from the fact that $H_iE' = H_iE'_i = -H_i\mathbf{1}\{I_i < 0\}I'_i = 0$. Again, comparing with (27) we see that $\frac{\partial IP^h_i}{\partial X_i} \geq (r-v-t)\frac{\partial T}{\partial X_i}$. Furthermore, for each sample path $\frac{\partial IP^h_i}{\partial X_i} \geq 0$. This implies that $\frac{\partial J^c}{\partial X_i} \leq \frac{\partial J^h_i}{\partial X_i}$.

Now consider AR-e which asserts the player i will get $\frac{E_i}{E}$ of the total profit $(r-v-t)T$. Therefore, the expected profit of player i is: $\mathbf{E}\{\frac{E_i}{E}[(r-v-t)T]$. That is, the indirect profit of player i is

$$\mathbf{E}[IP^e_i(\vec{X})] = (r-v-t)\mathbf{E}\{\frac{E_i}{E}T\}.$$

Suppose $H < E$. Then,

$$IP^e_i(\vec{X}) = (r-v-t)\frac{E_i}{E}H.$$

Taking derivative w.r.t. X_i, we get,

$$\begin{aligned}\frac{\partial IP^e}{\partial X_i} &= (r-v-t)\frac{E'_iHE_{-i}}{E^2}\\ &\leq (r-v-t)\frac{\partial T}{\partial X_i}.\end{aligned}$$

On the other hand, if $H > E$, then

$$IP^e_i(\vec{X}) = E_i.$$

Taking derivative w.r.t. X_i, we get,

$$\frac{\partial IP^e}{\partial X_i} = (r-v-t)E'_i = (r-v-t)\frac{\partial T}{\partial X_i}.$$

This implies that $\frac{\partial IP^e_i}{\partial X_i} \leq (r-v-t)\frac{\partial T}{\partial X_i}$ for each sample path and hence, $\frac{\partial J^e_i}{\partial X_i} \leq \frac{\partial J^c}{\partial X_i}$. Hence the result follows.

Proof of Theorem 4.2: Consider an allocation that chooses AR–h with probability $\gamma \in [0,1]$ and AR–e with probability $1-\gamma$. Denote this policy as AR–(e+h). Then the expected allocation of surplus to player i under this scheme is:

$$\begin{aligned}\mathbf{E}[IP^{e+h}_i(\vec{X}) &= (r-v-t)\mathbf{E}\left[\gamma\frac{H_i}{H}T + (1-\gamma)\frac{E_i}{E}T\right]\\ &= (r-v-t)\mathbf{E}_{H\leq E}\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]T + \mathbf{E}_{H>E}\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]T\\ &= (r-v-t)\mathbf{E}_{H\leq E}\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]H + \mathbf{E}_{H>E}\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]E\end{aligned}$$

Taking derivative w.r.t. X_i, we get

$$\begin{aligned}\frac{\partial\mathbf{E}[IP^{e+h}_i(\vec{X})]}{\partial X_i} &= (r-v-t)\mathbf{E}_{H\leq E}\frac{\partial}{\partial X_i}\left\{\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]H\right\}\\ &\quad +(r-v-t)\mathbf{E}_{H>E}\frac{\partial}{\partial X_i}\left\{\left[\gamma\frac{H_i}{H} + (1-\gamma)\frac{E_i}{E}\right]E\right\}\\ &= (r-v-t)\Big[\mathbf{E}_{H\leq E}[H'_i] + (1-\gamma)\mathbf{E}_{H\leq E}\left(\frac{E'_iHE_{-i}}{E^2} - H'_i\right)\Big]\\ &\quad +(r-v-t)\Big[\mathbf{E}_{H>E}[E'_i] + \gamma\int_{H>E}\left(\frac{H'_iEH_{-i}}{H^2} - E'_i\right)\Big]\end{aligned}$$

$$= (r-v-t)\mathbf{E}\left[\frac{\partial T}{\partial X_i}\right] + (r-v-t)\Big[(1-\gamma)\mathbf{E}_{H\le E}\left(\frac{E_i' H E_{-i}}{E^2} - H_i'\right) + \gamma\mathbf{E}_{H>E}\left(\frac{H_i' E H_{-i}}{H^2} - E_i'\right)\Big]$$

If we choose γ such that the second term above evaluated at $\vec{X}^c$ is equal to zero, we will have that

$$\frac{\partial \mathbf{E}[IP_i^{e+h}(\vec{X})]}{\partial X_i} = (r-v-t)\mathbf{E}\left[\frac{\partial T}{\partial X_i}\right]$$

and we will achieve the centralized solution. That is,

$$(1-\gamma)\mathbf{E}_{H^c\le E^c}\left(\frac{(E_i^c)' H^c E_{-i}^c}{(E^c)^2} - (H_i^c)'\right) + \gamma\mathbf{E}_{H^c>E^c}\left(\frac{(H_i^c)' E^c H_{-i}^c}{(H^c)^2} - (E_i^c)'\right)\Big] = 0$$

where H_i^c and E_i^c represent the excess stock and excess demand of player i when the inventory decision is $\vec{X}^{c*}$. Simplifying, we get

$$\gamma = \frac{\mathbf{E}_{H^c\le E^c}\left(\frac{(E_i^c)' H^c E_{-i}^c}{(E^c)^2} - (H_i^c)'\right)}{\mathbf{E}_{H^c\le E^c}\left(\frac{(E_i^c)' H^c E_{-i}^c}{(E^c)^2} - (H_i^c)'\right) - \mathbf{E}_{H^c>E^c}\left(\frac{(H_i^c)' E^c H_{-i}^c}{(H^c)^2} - (E_i^c)'\right)}$$

Recall that $E_i' \le 0$ and $H_i' \ge 0$, which implies the numerator and denominator in γ are non–positive and hence $\gamma \ge 0$. Furthermore, the absolute value of the denominator is larger than the absolute value of the numerator in γ and hence $\gamma \le 1$.

Now we need to show that this randomized policy is in the meta–core. Recall that the Nash equilibrium is symmetric. If the demands are Normally distributed, the inventory decision of each player in the grand coalition can be represented as $X = \mu + k_N\sigma$. Using (25), the allocation to each player can be written as

$$\frac{(N-\sqrt{N})(k_N\sigma\Phi(k_N) + \sigma\phi(k_N))}{N}.$$

Now consider any sub-coalition $\mathcal{S}$. The randomized policy applied to this sub-coalition will give the same solution as if a single decision maker optimized inventory for all players in the sub-coalition. Let the inventory decision of each player in this sub-coalition be $\tilde{X} = \mu + k_S\sigma$. Clearly $k_S \le k_N$. Then we have the following:

$$\begin{aligned}\frac{(N-\sqrt{N})(k_N\sigma\Phi(k_N) + \sigma\phi(k_N))}{N} &= (1-\frac{1}{\sqrt{N}})(k_N\sigma\Phi(k_N) + \sigma\phi(k_N)) \\ &> (1-\frac{1}{\sqrt{|\mathcal{S}|}})(k_N\sigma\Phi(k_N) + \sigma\phi(k_N)) \\ &\ge (1-\frac{1}{\sqrt{|\mathcal{S}|}})(k_S\sigma\Phi(k_S) + \sigma\phi(k_S)).\end{aligned}$$

∎

Proof of Proposition 4.3:

Since there is single warehouse with claimed inventory, we let Y_n denote claimed inventory of retailer n. For expositional clarity, we will write $W_{\mathcal{S}_n}^* = W^*(Y_1,\ldots,Y_{i-1})$. The payoff function of player n is then

$$J_n^{ic}(\vec{Y}) = W^*(Y_1,\ldots,Y_n) - W^*(Y_1,\ldots,Y_{n-1}),$$

This implies that

- the decision of retailer n is independent of the decisions, $Y_{n+1}, \ldots, Y_N$, of players $n+1, ..., N$.
- player n needs to optimize only $W^*(Y_1, \ldots, Y_n)$.

We now prove that the inventory in the centrally coordinated system is equal to the sum of inventories in the decentralized system with AR-ic. We begin by writing the profit function for the centralized system. Recall that all stock in the centralized system will be held at a single location. Let this stock be denoted by Y. Furthermore, define cumulative demands $D_{i,j} = D_i + \cdots + D_j$, for $j \geq i$ with cumulative density function as $F_{i,j}(\cdot)$. Observe that in the centralized system demands will be satisfied in the order of highest revenue first. The expected profit function in the centralized system is then:

$$\begin{aligned} J^c(Y) = -cY + r_N \mathbf{E}[\min\{D_N, Y\}] \\ + \cdots + r_i \mathbf{E}[\min\{D_i, \max\{Y - D_{i+1,N}, 0\}\}] \\ + \cdots + r_1 \mathbf{E}[\min\{D_1, \max\{Y - D_{2,N}, 0\}\}] \\ + v\mathbf{E}[\max\{Y - D_{1,N}, 0\}] \end{aligned}$$

Taking the derivative with respect to Y we get:

$$\frac{\partial J^c}{\partial Y} = -c + r_N(1 - F_N(Y)) + \cdots + r_i \int_{D_{i+1,N}=0}^{Y} (1 - F_i(Y - D_{i+1,N}) dF_{i+1,N}(D_{i+1,N})$$
$$+ \cdots + r_1 \int_{D_{2,N}=0}^{Y} (1 - F_1(Y - D_{2,N}) dF_{2,N}(D_{2,N}) + vF_{1,N}(Y).$$

Equating it to zero, we get the optimal inventory in the centralized system as Y^c.

To derive the expression for the first partial of the payoff function of player n, first notice that $W^*(Y_1, \cdots, Y_N) = J^c(Y)$, where $Y = \sum_{n=1}^{N} Y_n$. Now, consider player N. Recall

$$\frac{\partial J_N^{ic}(\vec{Y})}{\partial Y_N} = \frac{\partial W^*(Y_1, \cdots, Y_N)}{\partial Y_N} = \frac{\partial J^c(Y)}{\partial Y_N}.$$

Thus, it is optimal to choose Y_N such that $\sum_{n=1}^{N} Y_n = Y^c$. It is always possible to choose such an $Y_N \geq 0$ since it is clear that $\sum_{n=1}^{N-1} Y_n^{ic*} \leq Y^c$, where Y_n^{ic*} is the optimal inventory choice of player n under AR–ic. ∎

Proof of Proposition 4.4:

First we show that AR–ic is in the core of **AGE**. It can be shown that the allocation to player n,

$$\alpha_n = r\mu_n + (\sqrt{n} - \sqrt{(n-1)})\sigma\{rk - (r-v)[(k\Phi(k) + \phi(k)]\}$$

where Φ and ϕ are the normal standard CDF and PDF, and k satisfies: $\Phi(k) = \frac{r-c}{r-v}$.

We need to show that for every coalition $\mathcal{S}$ we have $\sum_{j \in \mathcal{S}} \alpha_j \geq W_{\mathcal{S}}^*$. The value of the coalition $\mathcal{S}$ is:

$$W_{\mathcal{S}}^* = |\mathcal{S}|^{0.5} \sigma\{rk - (r-v)(k\Phi(k) + \phi(k))\} + r \sum_{j \in S} \mu_j,$$

where $|\mathcal{S}|$ is the number of players in the coalition $\mathcal{S}$.

Thus,

$$\sum_{j\in\mathcal{S}}\alpha_j = r\sum_{j\in S}\mu_j + \sum_{j\in\mathcal{S}}(\sqrt{j}-\sqrt{(j-1)})\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}.$$

To prove $\sum_{j\in\mathcal{S}}\alpha_j \geq W^*_{\mathcal{S}}$, it is sufficient to show that

$$\sum_{j\in\mathcal{S}}(\sqrt{j}-\sqrt{(j-1)})\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\} \geq |\mathcal{S}|^{0.5}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}.$$

Since the expression $rk-(r-v)(k\Phi(k)+\phi(k))<0$ (this can be easily proved from the definition of k and the fact that $\phi(\cdot)$ is always positive) it is sufficient to prove that

$$\sum_{j\in\mathcal{S}}(\sqrt{j}-\sqrt{(j-1)}) \leq |\mathcal{S}|^{0.5}.$$

Observe that $\sqrt{j}-\sqrt{(j-1)}$ is decreasing in j. Thus, it is sufficient to prove that

$$\sum_{j=1}^{|\mathcal{S}|}(\sqrt{j}-\sqrt{(j-1)}) \leq |\mathcal{S}|^{0.5}.$$

But $\sum_{j=1}^{|\mathcal{S}|}(\sqrt{j}-\sqrt{(j-1)}) = |\mathcal{S}|^{0.5}$ which completes the proof that AR–ic is in the core of **AGE**.

Now we show that AR–ic is in the meta-core. We need to show that for every $\mathcal{S}$ and every allocation policy AR–β, $\sum_{j\in\mathcal{S}}\alpha_j \geq W^{\beta *}_{\mathcal{S}}$, the value of the coalition $\mathcal{S}$ under AR-β. Clearly $W^{\beta *}_{\mathcal{S}}$ is maximized whenever AR–β achieves the centralized solution for $\mathcal{S}$. But, it is straightforward to see that the marginal solution applied to the coalition $\mathcal{S}$ achieves the centralized solution. Let the allocation policy AR–ic be the marginal allocation policy applied to the grand coalition, and the allocation policy AR–$\widetilde{\text{ic}}$ be the marginal allocation policy applied to the coalition $\mathcal{S}$. We need to prove that $\sum_{j\in\mathcal{S}}\alpha_j \geq W^{\widetilde{ic}}_{\mathcal{S}} = \sum_{j\in\mathcal{S}}\widetilde{\alpha}_j$. It is sufficient to show that $\alpha_j \geq \widetilde{\alpha}_j \quad \forall j \in \mathcal{S}$. Let the players in $\mathcal{S}$ be ordered in increased order of revenues; that is, $r_{i_1} \leq r_{i_2} \leq \cdots \leq r_{i_{|\mathcal{S}|}}$. Let S_{i_k} be the set of all players $k \neq i_k$ in the grand coalition such that $r_k \leq r_{i_k}$. Clearly,

$$\alpha_{i_1} = W^{ic}_{S_{i_1}\cup\{i_1\}} - W^{ic}_{S_{i_1}} \geq W^{ic}_{\{i_1\}} = W^{\widetilde{ic}}_{\{i_1\}} = \widetilde{\alpha}_{i_1}.$$

Similarly,

$$\alpha_{i_2} = W^{ic}_{S_{i_2}\cup\{i_2\}} - W^{ic}_{S_{i_2}} = W^{ic}_{\widehat{S}_{i_2}\cup\{i_1\}\cup\{i_2\}} - W^{ic}_{\widehat{S}_{i_2}\cup i_1},$$

where $\widehat{S}_{i_2} = S_{i_2}\backslash\{i_1\}$.

Notice that,

$$\begin{aligned}\alpha_2 &= W^{ic}_{\widehat{S}_{i_2}\cup\{i_1\}\cup\{i_2\}} - W^{ic}_{\widehat{S}_{i_2}\cup\{i_1\}}\\ &= (|\widehat{S}_{i_2}|+2)^{0.5}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\} + r\sum_{i\in\widehat{S}_{i_2}\cup\{i_1\}\cup\{i_2\}}\mu\end{aligned}$$

$$
\begin{aligned}
&-\{(|\widehat{S}_{i_2}|+1)^{0.5}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\sum_{i\in\widehat{S}_{i_2}\cup\{i_1\}}\mu\}\\
&=\{(|\widehat{S}_{i_2}|+2)^{0.5}-(|\widehat{S}_{i_2}|+1)^{0.5}\}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\mu\\
&>(2^{0.5}-1)\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\mu\\
&=\widetilde{\alpha}_2
\end{aligned}
$$

Proceeding to generalize for the k player in $\mathcal{S}$, we write:

$$
\begin{aligned}
\alpha_k &= W^{ic}_{\hat{S}_{i_k}\cup\{i_1\}\cup\cdots\cup\{i_k\}} - W^{ic}_{\hat{S}_{i_k}\cup\{i_1\}\cup\cdots\cup\{i_{k-1}\}}\\
&= (|\hat{S}_{i_k}+k)^{0.5}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\sum_{i\in\widehat{S}_{i_k}\cup\{i_1\}\cup\cdots\cup\{i_k\}}\mu\\
&\qquad -(|\hat{S}_{i_k}+k-1)^{0.5}\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\sum_{i\in\widehat{S}_{i_k}\cup\{i_1\}\cup\cdots\cup\{i_{k-1}\}}\mu\\
&> (k^{0.5}-(k-1)^{0.5})\sigma\{rk-(r-v)(k\Phi(k)+\phi(k))\}+r\mu\\
&= \widetilde{\alpha}_k.
\end{aligned}
$$

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